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THE

MESSENGER OF MATHEMATICS.

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CONTENTS.

ARITHMETIC AND ALGEBRA.

PAGE

A new proof that a general quadric may be reduced to its canonical form (*i.e.* a linear function of squares) by means of a real orthogonal substitution. By Prof. J. J. Sylvester - - - - 1

Note on the sums of two series. By Prof. Cayley - - - - 29

On the reduction of a bilinear quantic of the *n*th order to the form of a sum of *n* products by a double orthogonal substitution. By Prof. J. J. Sylvester - - - - - - - - 43

On an arithmetical theorem in periodic continued fractions. By Prof. J. J. Sylvester - - - - - - - 63

Notes on modern higher algebra. By T. S. Fiske - - - - 89

Gordan's series. By H. F. Baker - - - - - - 91

Solution of $(a, b, \ldots, c) = (a^p, b^p, \ldots, c^p)$. By H. W. Lloyd Tanner - 118

On Latin squares. By Prof. Cayley - - - - - - 135

Note on series whose coefficients involve powers of the Bernoullian numbers By J. W. L. Glaisher - - - - - - - 138

A proof of the theorem of reciprocity for quadratic residues. By F. Franklin - - - - - - - - 176

On the equation $x^{17} - 1 = 0$. By Prof. Cayley - - - - 184

Some inequalities. By W. Segar - - - - - - 189

GEOMETRY.

On the converses of some theorems in elementary geometry. By C. Leudesdorf - - - - - - - 14

The solution of a special case of the problem of the establishment of a correlation between two figures. By J. Brill - - - 57

On an extension of a problem of Pappus's. By Isuruta Kenji - - 67

Geometrical interpretation of a condition of integrability. By W. Burnside 96

The lines of zero length on a surface as curvilinear coordinates. By W. Burnside - - - - - - - - 99

Note on the intersection of lines with curves of an odd degree. By H. G. Dawson - - - - - - - 110

On the focals of a quadric surface. By Prof. CAYLEY - - - 113

The solution of a special case of the problem of the establishment of a correlation between two plane figures (second paper). By J. BRILL - 151

Note on reciprocal lines. By Prof. CAYLEY - - - - - 174

Note de géométrie, à propos d'un théorème de M. Stewart. By Prof. MANNHEIM - - - - - - - - 178

Quaternion proofs of theorems relating to asymptotic lines. By T. MOTODA 188

DIFFERENTIAL AND INTEGRAL CALCULUS, AND FINITE DIFFERENCES.

Note on the differential equation of a conic. By E. B. ELLIOTT - - 5

On differential expressions which persist in form after the transformation $x=\frac{1}{x'}$, $y=\frac{y'}{x'}$. By E. B. ELLIOTT - - - - - 7

On Stirling's formula and other interpolation formulæ. By G. H. STUART - 19

THEORY OF ELLIPTIC FUNCTIONS.

Note on the sums of two series. By Prof. CAYLEY - - - - 29

Expansions of K, I, G, E in powers of k'^2-k^2. By J. W. L. GLAISHER - 146

On a direct relation between the definite elliptic integrals of the first and second order. By G. F. CHILDE - - - - - 155

On the expansion of $\frac{G}{K}$, $\frac{K}{G}$, $\frac{I}{E}$, &c., in ascending powers of k^2. By J. W. L. GLAISHER - - - - - - - 164

APPLIED MATHEMATICS.

Note on energy in an elastic solid. By KARL PEARSON - - - 31

The parabolic trajectory. By Prof. GREENHILL - - - - 47

The small deformation of curves and surfaces with application to the vibrations of a helix and a circular ring. By J. H. MICHELL - - 68

On the exhaustion of Neumann's mode of solution for the motion of solids of revolution in liquids, and similar problems. By J. H. MICHELL - 83

Vibrations of a string stretched on a surface. By J. H. MICHELL - - 87

Propagation of energy in the electro-magnetic field. By W. BURNSIDE - 98

On the resultant of two finite displacements of a rigid body. By W. BURNSIDE - - - - - - - - 102

Note on Clapeyron's theorem of the three moments. By KARL PEARSON - 129

On the stability of a bent and twisted wire. By J. H. MICHELL - - 181

95

MESSENGER OF MATHEMATICS.

A NEW PROOF THAT A GENERAL QUADRIC MAY BE REDUCED TO ITS CANONICAL FORM (THAT IS, A LINEAR FUNCTION OF SQUARES) BY MEANS OF A REAL ORTHOGONAL SUBSTITUTION.

By Prof. *J. J. Sylvester.*

All the proofs that I am acquainted with (and their name is legion) of the possibility of depriving a quadric, in three or more variables, of its mixed terms by a real orthogonal transformation are made to depend on the theorem that the "*latent roots*" of any symmetrical matrix are all real.

By the latent roots is understood the roots of the determinant expressed by tacking on a variable $-\lambda$ to each term in the diagonal of symmetry to such matrix.

I shall show that the same conclusion may be established *à priori* by purely algebraical ratiocination and without constructing any equation, by the method of cumulative variation. The proof I employ is inductive: *i.e.* if the theorem is true for two or any number of variables I prove that it will be true for one more.

To illustrate the method let us begin with two variables. Consider the form $ax^2 + 2hxy + by^2$.

If in any such form $b = a$, then by an obvious orthogonal transformation, viz. writing $\dfrac{x+y}{\sqrt{(2)}}$ and $\dfrac{x-y}{\sqrt{(2)}}$ for x and y, the the form becomes

$$a(x^2+y^2) + h(x^2-y^2),$$

or

$$(a+h)x^2 + (a-h)y^2.$$

Now in general on imposing on x, y any orthogonal infinitesimal substitution, so that

$$\begin{array}{lcl} x & \text{becomes} & x+\varepsilon y, \\ y & ,, & y-\varepsilon x \end{array}$$

h in the new form becomes $h+(a-b)\varepsilon$, or say $\delta h=(a-b)\varepsilon$, and $\frac{1}{2}\delta(h^2)=(a-b)h\varepsilon$; the variations of a and b need not be set forth.

Let an infinite succession of such transformations be instituted; then either a and b become equal and the orthogonal substitution above referred to reduces the quadric to its canonical form [in which case this one combined with the preceding infinite series of such substitutions may be compounded into a single substitution], or else by giving ε the sign of $(b-a)$ the variation of h^2 may at each step be made negative so that h^2 continually decreases, unless h vanishes. If h does not vanish it must have a minimum value, and this minimum value may be diminished, which involves a contradiction; hence, in the infinite series of substitutions supposed, either a and b become equal or h vanishes, and in either case the quadric is reduced or reducible to its canonical form.

Let us now take the case of three variables x, y, z.

Obviously, by the preceding case, we may make the term involving xy disappear and commence with the initial form

$$ax^2+by^2+2fxz+2gyz+cz^2.$$

If f or g become zero the quadric may be canonified by virtue of the preceding case.

Again, if $b=a$ by imposing on x, y the orthogonal substitution

$$\frac{g}{\sqrt{(f^2+g^2)}}x+\frac{f}{\sqrt{(f^2+g^2)}}y$$

$$-\frac{f}{\sqrt{(f^2+g^2)}}x+\frac{g}{\sqrt{(f^2+g^2)}}y,$$

the term involving xz will disappear and the final result is the same as if f were zero.

Let us now introduce the infinitesimal orthogonal substitution which changes

$$\begin{array}{lcl} x & \text{into} & x+\varepsilon y+\eta z, \\ y & ,, & -\varepsilon x+y+\theta z, \\ z & ,, & -\eta x-\theta y+z, \end{array}$$

where ε, η, θ are supposed to be of the same order of magnitude so that only first powers of them have to be considered.

Then
$$\delta f = (a-c)\,\eta - g\varepsilon,$$
$$\delta g = (b-c)\,\theta + f\varepsilon,$$
also the coefficient of $2xy$ becomes $(a-b)\,\varepsilon - f\theta - g\eta$.

Now whatever η, θ may be, we may determine ε in terms of η, θ so that this may be made to vanish, and the initial form of the quadric will be maintained, provided that b is not equal to a.

Hence instituting an infinite series of these infinitesimal substitutions, provided we do not reach a stage where a and b become equal, we may maintain the original form keeping η, θ arbitrary, and shall have
$$\tfrac{1}{2}\delta\,(f^2+g^2) = (a-c)\,f\eta + (b-c)\,g\theta.$$

Suppose a and b to be unequal; therefore $(a-c)$, $(b-c)$ do not vanish simultaneously, and consequently we may make $\delta\,(f^2+g^2)$ negative unless at least one of the two quantities f, g vanishes.

If neither of them vanishes f^2+g^2 may be made continually to decrease and will have a minimum other than zero, which involves a contradiction.

Hence the infinite series of infinitesimal orthogonal substitutions may be so conducted that either $a-b$ or one at least of the letters f, g shall become zero; and then two additional orthogonal substitutions at most will serve to reduce the Quadric immediately to its canonical form.

I shall go one step further to the case of four variables x, y, z, t, and then the course of the induction will become manifest. We may, by virtue of what has been shown, take as our quadric
$$ax^2 + by^2 + cz^2 + 2fxt + 2gyt + 2hzt + dt^2.$$

Here, if any one of the mixed terms disappears, the quadric is immediately reducible by the preceding case, and if any two of the grouped pure coefficients a, b, c become equal (as for instance a, b), then by an orthogonal transformation one of the mixed terms (f or g in the case supposed) may be got rid of; so that this supposition merges in the preceding one.

Impose on x, y, z, t an infinitesimal orthogonal substitution, writing
$$\begin{array}{rl}
x + \varepsilon y + \theta z + \lambda t & \text{for } x,\\
-\varepsilon x + y + \eta z + \mu t & \quad ,, \quad y,\\
-\theta x - \eta y + z + \nu t & \quad ,, \quad z,\\
-\lambda x - \mu y - \nu z + t & \quad ,, \quad t.
\end{array}$$

Then
$$\begin{aligned}\delta f &= (a-d)\,\lambda - g\varepsilon - h\theta,\\ \delta g &= (b-d)\,\mu + f\varepsilon - h\eta,\\ \delta h &= (c-d)\,\nu + f\theta + g\eta.\end{aligned}$$

Also the coefficients of $2xy$, $2xz$, $2yz$ respectively become
$$\begin{aligned}&(a-b)\,\varepsilon - f\mu - g\lambda,\\ &(a-c)\,\theta - f\nu - h\lambda,\\ &(b-c)\,\eta - g\nu - h\mu.\end{aligned}$$

Suppose that no two of the grouped pure coefficients a, b, c are equal; then ε, θ, η can be, and are to be, expressed in terms of λ, μ, ν so as to make these three expressions vanish; that being done the initial form of the Quadric is maintained throughout the series of substitutions and we may write
$$f\delta f + g\delta g + h\delta h = (a-d)f\lambda + (b-d)\,g\mu + (c-d)\,h\nu.$$

Of the three quantities λ, μ, ν it is sufficient for the purpose of the argument to retain any two as λ, μ and to suppose $\nu = 0$.

Then, since we suppose that a and b are not equal,
$$(a-d)f\lambda + (b-d)\,g\mu$$
(where λ, μ are arbitrary) can always be made negative unless f, g are one of them zero; so that if a and b never become equal *nor* f or g vanish $f^2 + g^2 + h^2$ cannot have any minimum value other than zero, which involves a contradiction; hence in the course of the series of infinitesimal transformations either a and b must become equal, or f or g or both of them vanish. If f and g vanish simultaneously or even if one only of them vanish, then *one* succeeding substitution, and if a and b become equal *two* succeeding substitutions will effect the reduction to the canonical form. This proves the theorem for four variables.

The method is obviously extendible to any number of variables; in the case just considered it is seen that in the infinitessmal orthogonal matrix of substitution the *exceptional* line or column [that which relates to the *excepted* variable the t] it is not necessary to employ more than two arbitrary infinitessimals and a like remark applies to the general case, so that if there are n variables whilst $\frac{1}{2}(n^2 - n)$ is the number of infinitessimals, that would appear in the complete matrix, $\frac{1}{2}(n^2 - 3n + 6)$ [*i.e.* $\frac{1}{2}\{(n-1)(n-2)\} + 2$], are sufficient for the purpose of the demonstration.

Thus then without recourse to any theorem of Equations it is proved that any Quadric may be reduced by a Real orthogonal substitution to its canonical form.*

New College, Oxford,
March 6, 1889.

NOTE ON THE DIFFERENTIAL EQUATION OF A CONIC.

By *E. B. Elliott.*

It is a surprisingly prevalent idea that the integration of

$$y''^2 y^{\mathrm{v}} - 5y''y'''y^{\mathrm{iv}} + \tfrac{40}{9}y'''^3 = 0,$$

the differential equation of a conic, is a matter of considerable difficulty. The recent papers† of Colonel Cunningham on the depression of differential equations must have done much to dispel the notion; but his processes of depression and integration of the equation occupy, as he says, some considerable space when written at length. It may be worth while then to show how simple the matter really is by writing down two variations of a perfectly direct and elementary method of performing the integration.

As the first two terms of the three on the left-hand side of the equation are multiples of the two parts of the result of differentiating $y''^2 y^{\mathrm{iv}}$, and the second and third of the two parts of that of differentiating $y''y'''^2$, it is clear that there must be an integrating factor of the form y''^m. Now upon multiplication by y''^m the equation may be written

$$y''^{m+2}y^{\mathrm{v}} + (m+2)\,y''^{m+1}y'''y^{\mathrm{iv}} - (m+7)\,y''^{m+1}y'''y^{\mathrm{iv}} + \tfrac{40}{9}y''^m y'''^3 = 0.$$

The first two terms are here an exact differential. The last two will be also an exact differential if

$$\frac{m+7}{2} + \frac{40}{9\,(m+1)} = 0,$$

---

* I have applied the same method to prove that by two real independent orthogonal substitutions operated on

$$x_1,\ x_2,\ \ldots,\ x_n;\quad y_1,\ y_2,\ \ldots,\ y_n$$

the general lineo-linear Quantic in the x's and y's (with real coefficients) may be reduced to the canonical form $\Sigma\,(x_i, y_i)$, and have sent for insertion in the C. R. of the Institute a Note in which I give the rule for effecting this reduction.

It may be sufficient here to mention that if U is the given lineo-linear Quantic, its n canonical multipliers are the square roots of the n canonical multipliers of the Quadric $\Sigma\left(\dfrac{dU}{dy}\right)^2$ $\left[\text{or if we please of } \Sigma\left(\dfrac{dU}{dx}\right)^2\right]$, which it may easily be shewn *à posteriori* are necessarily omni-positive; and I need hardly add that although these two Quadrics are different, their canonical multipliers are the same.

† Vol. XVII., pp. 118–145, Vol. XVIII., pp. 122–127.

i.e. if $m = -\frac{11}{3}$ or $-\frac{13}{3}$. Thus there are two integrating factors, $y''^{-\frac{11}{3}}$ and $y''^{-\frac{13}{3}}$. Either of these leads to a complete integration having no difficulty whatever.

I. The factor $y''^{-\frac{11}{3}}$ gives the equation the form

$$\frac{d}{dx}(y''^{-\frac{5}{3}}y^{iv}) - \tfrac{5}{3}\frac{d}{dx}(y''^{-\frac{8}{3}}y'''^2) = 0.$$

Hence at once comes the first integral

$$\frac{y''y^{iv} - \frac{5}{3}y'''^2}{y''^{\frac{8}{3}}} = C,$$

the second

$$\frac{y'''}{y''^{\frac{5}{3}}} = Cx + C',$$

and the third

$$\frac{1}{y''^{\frac{2}{3}}} = -\tfrac{2}{3}(\tfrac{1}{2}Cx^2 + C'x) + C'',$$

i.e.
$$y'' = \frac{1}{(Ax^2 + 2Bx + C)^{\frac{3}{2}}}.$$

One more integration gives

$$y' + \alpha = \frac{1}{AC - B^2}\frac{Ax + B}{(Ax^2 + 2Bx + C)^{\frac{1}{2}}},$$

and another gives the complete primitive

$$y + \alpha x + \beta = \frac{1}{AC - B^2}(Ax^2 + 2Bx + C)^{\frac{1}{2}},$$

i.e.
$$(AC - B^2)^2(y + \alpha x + \beta)^2 = Ax^2 + 2Bx + C,$$

a form of the general equation of the second degree.

II. Almost as simple is the use of the other integrating factor $y''^{-\frac{13}{3}}$. It gives the equation the form

$$\frac{d}{dx}(y''^{-\frac{7}{3}}y^{iv}) - \tfrac{4}{3}\frac{d}{dx}(y''^{-\frac{10}{3}}y'''^2) = 0,$$

whose first integral is

$$\frac{y''y^{iv} - \frac{4}{3}y'''^2}{y''^{\frac{10}{3}}} = c,$$

i.e.
$$\frac{y''y^{iv} - \frac{4}{3}y'''^2}{y''^{\frac{7}{3}}} = cy'',$$

an exact differential equation which yields upon integration,

$$\frac{y'''}{y''^{\frac{4}{3}}} = cy' + c',$$

which again leads at once to

$$\frac{1}{y''^{\frac{1}{3}}} = -\tfrac{1}{3}(cy + c'x) + c'',$$

or, say,
$$y'' = (Ax + By + C)^{-3},$$
i.e.
$$u'' = Bu^{-3},$$
where
$$u = Ax + By + C.$$

The immediate integral of this is

$$u'^2 = -B(u^{-2} - C'),$$

i.e.
$$\frac{uu'}{\sqrt{(u^2 - C'^{-1})}} = \sqrt{(BC')},$$

whence at once the complete primitive

$$\sqrt{(u^2 - C'^{-1})} = 2\sqrt{(BC')}\,x + C'',$$

which, by reintroduction of the value of u and alteration of the constants, is the same as

$$(Ax + By + C)^2 - D = (Ex + F)^2,$$

and is the general equation of the second degree.

Queen's College, Oxford,
February, 1889.

ON DIFFERENTIAL EXPRESSIONS WHICH PERSIST IN FORM AFTER THE TRANSFORMATION $x = \frac{1}{x'}$, $y = \frac{y'}{x'}$.

By *E. B. Elliott.*

In a paper which I have recently been preparing for the *London Mathematical Society* I have been led to discuss the formation of a complete system of functions of x, y, z and the partial differential coefficients of x with regard to y and z, which persist in form after the transformation

$$\frac{x}{x'} = \frac{y}{y'} = \frac{z}{1} = \frac{1}{z'}.$$

The analogous treatment of the simpler question of persistents for the transformation of two variables only

$$\frac{x}{1}=\frac{1}{x'}=\frac{y}{y'}$$

may have interest, even though most of the results which can be derived from its conclusions have been obtained previously by others who have considered the transformation. The idea of making everything depend on the choice of a complete system of linear persistents is I believe new, and the facility of determination and simplicity of form of such a complete system is perhaps remarkable.

The formulæ of transformation are equivalent to the pair

$$\log x = -\log x' = u, \text{ say,}$$
$$x^{-\frac{1}{2}}y = \quad x'^{-\frac{1}{2}}y' = v, \text{ say;}$$

which tell us that u and v are functions which persistent in form after the transformation, the latter persisting absolutely, and the former but for a change of sign. This may be expressed by saying that v is a persistent of positive and u one of skew or negative character.

The expressions for the old and new variables in terms of the persistent functions u and v are

$$x=e^u,\ y=ve^{\frac{1}{2}u},\ x'=e^{-u},\ y'=ve^{-\frac{1}{2}u},$$

so that any function of x and y is capable of unique expression in terms of u and v, the same function of x' and y' having a value which is obtained from that of the function of x and y by altering the sign of u. Persistent functions of x and y are then those whose expressions in terms of u and v are unaltered or changed only in sign by writing $-u$ for u.

Now it is supposed that a relation, perfectly unrestricted in form, connects x and y. The same may be expressed as a relation connecting u and v. We may regard then v as a function of u just as we may regard y as a function of x.

The values of u in terms of x and x' give us the equivalence of operators

$$x\frac{d}{dx}=\frac{d}{du}=-x'\frac{d}{dx'},$$

the symbols being those of total differentiation, so that, for instance, if the function operated on be expressed as one of x and y, $\frac{d}{dx}$ means $\left(\frac{d}{dx}\right)+\frac{dy}{dx}\left(\frac{d}{dy}\right)$. We may express this equivalence of operators by saying that $\frac{d}{du}$ is a persistent operator of negative character.

It follows that, for every positive integral value of n, $\frac{d^n v}{du^n}$ is a persistent function; in fact that

$$\left(x\frac{d}{dx}\right)^n (x^{-\frac{1}{2}}y) = \frac{d^n v}{du^n} = (-1)^n \left(x'\frac{d}{dx'}\right)^n (x'^{-\frac{1}{2}}y').$$

The persistent $\frac{d^n v}{du^n}$ is thus of positive or negative character according as n is even or odd.

In u, v, $\frac{dv}{du}$, $\frac{d^2 v}{du^2}$, ..., $\frac{d^n v}{du^n}$, we have a system of independent persistent functions in the arguments x, y, $\frac{dy}{dx}$, $\frac{d^2 y}{dx^2}$, ..., $\frac{d^n y}{dx^n}$ which is complete up to the n^{th} order. This is clear, for there are just as many persistents in the system as there are arguments, and each involves an argument which does not occur in any of the preceding. Were there any other persistent in the arguments which was not dependent on these, the $n+2$ quantities x', y', $\frac{dy'}{dx'}$, ..., $\frac{d^n y'}{dx'^n}$ could be chosen so as to satisfy $n+3$ equations quite independent of the supposed relation which connects x' and y'; and this is impossible.

Moreover the persistents v, $\frac{dv}{du}$, $\frac{d^2 v}{du^2}$, ..., are all linear in y, $\frac{dy}{dx}$, $\frac{d^2 y}{dx^2}$, ..., the coefficients being functions of x. Thus the complete aggregate of persistent functions is involved in a simple system of linear persistents.

If we denote $\frac{1}{n!}\frac{d^n y}{dx^n}$ by y_n the early persistents of this linear system are

$$v = x^{-\frac{1}{2}}y = +$$

$$\frac{dv}{du} = x^{\frac{1}{2}}y_1 - \tfrac{1}{2}x^{-\frac{1}{2}}y = -$$

$$\frac{1}{2!}\frac{d^2 v}{du^2} = x^{\frac{3}{2}}y_2 + \tfrac{1}{8}x^{-\frac{1}{2}}y = +$$

$$\frac{1}{3!}\frac{d^3 v}{du^3} = x^{\frac{5}{2}}y_3 + \tfrac{1}{2}x^{\frac{3}{2}}y_2 + \tfrac{1}{24}x^{\frac{1}{2}}y_1 - \tfrac{1}{48}x^{-\frac{1}{2}}y = -$$

$$\frac{1}{4!}\frac{d^4 v}{du^4} = x^{\frac{7}{2}}y_4 + x^{\frac{5}{2}}y_2 + \tfrac{5}{24}x^{\frac{3}{2}}y_2 + \tfrac{1}{384}x^{-\frac{1}{2}}y = +,$$

&c., &c., the notation on the right denoting that each function on the left is equal to the same function of the accented letters taken positively or negatively as in its particular case indicated.

It is at once evident that the system of linear persistents here obtained is not the simplest complete system. For instance, by subtraction from the third of one-eighth of the first we obtain the simplest persistent of the second order

$$x^{\frac{3}{2}}y_2 = + ;$$

by subtraction of one twenty-fourth of the second from the fourth we obtain in like manner a simplified persistent of the third order

$$x^{\frac{5}{2}}y_3 + \tfrac{1}{2}x^{\frac{3}{2}}y_2 = - ;$$

by subtraction of multiples of the first and third from the fifth we get

$$x^{\frac{7}{2}}y_4 + x^{\frac{5}{2}}y_3 = + ;$$

&c., &c.

That there is a complete linear system, from the second order onwards, not involving either y or the first derivative y_1, is now obvious. For in $x^{\frac{3}{2}}y_2$ we have such a linear persistent of the second order, and by no number of operations with $x\dfrac{d}{dx}$ or $\dfrac{d}{du}$ can a function involving y or y_1 be obtained from a function which is free from those arguments.

It will now be shown that the type of a complete system of linear persistents free from the dependent variable and its first derivative is, from the second order onwards,

$$v_n = x^{\frac{1}{2}(2n-1)}\left\{y_n + \frac{n-2}{2x}y_{n-1} + \frac{(n-2)(n-3)}{(2x)^2\,2\,!}y_{n-2}\right.$$
$$\left. + \frac{(n-2)(n-3)(n-4)}{(2x)^3\,3\,!}y_{n-3} + \dots + \frac{1}{(2x)^{n-2}}y_2\right\},$$

which may be written symbolically in several compact forms,

e.g.
$$v_n = x^{\frac{1}{2}(2n-1)}\left(E + \frac{1}{2x}\right)^{n-2}y_2$$
$$= x^{\frac{1}{2}(2n-1)}\left(1 + \frac{1}{2x}E^{-1}\right)^{n-2}y_n,$$

where E is the operator which converts y_r into y_{r+1}, and E^{-1} that which converts y_{r+1} into y_r, *i.e.* where

$$E\ =y_3\frac{d}{dy_2}+y_4\frac{d}{dy_3}+y_5\frac{d}{dy_4}+\ldots,$$

$$E^{-1}=y_2\frac{d}{dy_3}+y_3\frac{d}{dy_4}+y_4\frac{d}{dy_5}+\ldots.$$

The form, however, which is of incomparably the greatest importance is

$$v_n=x^{\frac{1}{2}(2n-1)}\,e^{\frac{1}{2x}\Omega}\,y_n,$$

where $$\Omega=y_2\frac{d}{dy_3}+2y_3\frac{d}{dy_4}+3y_4\frac{d}{dy_5}+\ldots.$$

The theorem now before us will be proved by establishing that

$$\frac{dv_n}{du}=(n+1)\,v_{n+1}-\tfrac{1}{4}\,(n-2)\,v_{n-1},$$

and basing upon it a Mathematical Induction.

By operating on the expanded form of v_n with $\dfrac{d}{du}$ or $x\dfrac{d}{dx}$, we have

$$\begin{aligned}\frac{dv_n}{du}=\tfrac{1}{2}\,(2n-1)\,x^{\frac{1}{2}(2n-1)}\Bigg\{&y_n+\frac{1}{2x}\,C_1^{n-2}y_{n-1}\\&+\frac{1}{(2x)^2}\,C_2^{n-2}y_{n-2}+\ldots+\frac{1}{(2x)^{n-3}}\,C_{n-3}^{n-2}y_3+\frac{1}{(2x)^{n-2}}\,y_2\Bigg\}\\&-x^{\frac{1}{2}(2n-1)}\Bigg\{\qquad\frac{1}{2x}\,C_1^{n-2}y_{n-1}\\&+\frac{2}{(2x)^2}\,C_2^{n-2}y_{n-2}+\ldots+\frac{n-3}{(2x)^{n-3}}\,C_{n-3}^{n-2}y_3+\frac{n-2}{(2x)^{n-2}}\,y_2\Bigg\}\\&+x^{\frac{1}{2}(2n+1)}\,(n+1)\,y_{n+1}\\&+x^{\frac{1}{2}(2n-1)}\Bigg\{n\,C_1^{n-2}y_n+\frac{n-1}{2x}\,C_2^{n-2}y_{n-1}\\&+\frac{n-2}{(2x)^2}\,C_3^{n-2}y_{n-2}+\ldots+\frac{3}{(2x)^{n-3}}\,y_3\Bigg\},\end{aligned}$$

where C_r^{n-2} denotes the number of combinations of $n-2$ things r together.

The multiplier in this sum of $\frac{1}{2^r}x^{\frac{1}{2}(2n-1-2r)}y_{n-r}$ is, for all values of r from 0 to $n-2$,

$$\tfrac{1}{2}(2n-1)\,C_r^{n-2}-r\,C_r^{n-2}+(n-r)\,C_{r+1}^{n-2}$$

which is equal to

$$\frac{(n-2)(n-3)\dots(n-r-1)}{2(r+1)!}$$
$$\times\{(2n-1)(r+1)-2r(r+1)+(n-r)(n-r-2)\}$$
$$=\frac{(n-2)(n-3)\dots(n-r-1)}{2(r+1)!}\{(n+1)(n-1)-r(r+1)\}$$
$$=\tfrac{1}{2}(n+1)\,C_{r+1}^{n-1}-\tfrac{1}{2}(n-2)\,C_{r-1}^{n-3}.$$

Hence it is seen that

$$\frac{dv_n}{du}=(n+1)\,x^{\frac{1}{2}(2n+1)}\left\{y_{n+1}+\frac{1}{2x}\,C_1^{n-1}y_{n-1}\right.$$
$$\left.+\frac{1}{(2x)^2}\,C_2^{n-1}y_{n-2}+\dots+\frac{1}{(2x)^{n-1}}\,y_2\right\}$$
$$-\tfrac{1}{4}(n-2)\,x^{\frac{1}{2}(2n-3)}\left\{y_{n-1}+\frac{1}{2x}\,C_1^{n-3}y_{n-2}\right.$$
$$\left.+\frac{1}{(2x)^2}\,C_2^{n-3}y_{n-3}+\dots+\frac{1}{(2x)^{n-3}}\,y_2\right\}$$
$$=(n+1)\,v_{n+1}-\tfrac{1}{4}(n-2)\,v_{n-1},$$

as was to be proved.

Now if v_{n-1} and v_n are persistents of opposite characters, v_{n-1} and $\frac{dv_n}{du}$ are two of the same character. The relation established tells us therefore that v_{n+1} is in the same case a persistent of the same character as v_{n-1}.

But $$v_2=x^{\frac{3}{2}}y_2,$$

and $$v_3=x^{\frac{5}{2}}\left(y_3+\frac{1}{2x}\,y_2\right),$$

have been seen to be persistents of positive and negative characters respectively. Accordingly it follows that

$$v_4=x^{\frac{7}{2}}\left(y_4+\frac{1}{x}\,y_3+\frac{1}{4x^2}\,y_2\right)$$

is a persistent of positive character—its first two terms alone are such as has been seen earlier—and consequently that

v_5, v_6, v_7, ..., in succession are persistents of alternately negative and positive character. Thus quite generally v_n is a persistent whose mark of character is $(-1)^n$.

These linear persistents v_n are all independent and constitute a complete system from the second order onwards in virtue of the same reasoning as that applied already to the linear system $\frac{d^n v}{du^n}$, of which they are certain leading terms.

As to non-linear persistents only a very little need be added. All persistents which do not involve y and y_1 are functions of $v_2, v_3, v_4, \ldots$, and u or $\log x$. Moreover, they have to be unaltered or changed only in sign if $v_3, v_5, v_7, \ldots$, and $\log x$ have their signs reversed. Any persistent whatever is a function of the above arguments together with v and $\frac{dv}{du}$.

Now it is a well-known theorem that if ϑ be a linear differential operator, and ϕ, ψ be two functions of the arguments on which it operates,

$$e^{\vartheta}\phi \, e^{\vartheta}\psi = e^{\vartheta}(\phi\psi).$$

Hence, remarking that Ω contains no symbol of partial differentiation with regard to x, we obtain

$$\begin{aligned} v_m v_n &= x^{\frac{1}{2}(2m-1)} e^{\frac{1}{2x}\Omega} y_m . x^{\frac{1}{2}(2n-1)} e^{\frac{1}{2x}\Omega} y_n \\ &= x^{\frac{1}{2}(2m+2n-2)} e^{\frac{1}{2x}\Omega} y_m . e^{\frac{1}{2x}\Omega} y_n \\ &= x^{\frac{1}{2}(2m+2n-2)} e^{\frac{1}{2x}\Omega} (y_m y_n); \end{aligned}$$

and similarly for the product of three or any number of linear persistents, the power of x occurring as a factor being in the general case $x^{\frac{1}{2}(2\Sigma m - i)}$, where i is the number of factors.

Suppose then that $H_{iw}(y_2, y_3, y_4, \ldots)$ is a function of the second and higher derivatives which is homogeneous (of degree i) and isobaric (of constant sum of suffixes w), and that $H_{iw}(v_2, v_3, v_4, \ldots)$ is the same function of the corresponding linear persistents. It follows that

$$H_{iw}(v_2, v_3, v_4, \ldots) = x^{\frac{1}{2}(2w-i)} e^{\frac{1}{2x}\Omega} H_{iw}(y_2, y_3, y_4, \ldots).$$

By this means we obtain a complete system of persistents of any degree and weight. The mark of character of any such persistent is $(-1)^w$.

Interesting results may be obtained from the equivalence just proved in connexion with Sylvester's and Halphen's formula (*Am. J.*, Vol. IX., p. 297)

$$H_{iw}(y_2', y_3', y_4', \ldots) = (-1)^w x^{2w-i} e^{\frac{1}{x}\hat{\Omega}} H_{iw}(y_2, y_3, y_4, \ldots).$$

The further application of my results would be to a very great extent identical in its conclusions with those developed by Sylvester from the formula here quoted. I confine myself therefore to the remark, that a pure persistent, or homogeneous isobaric function of the second and higher derivatives only which persists, but for a power of x as factor, after our transformation, is exactly $x^{-\frac{1}{2}(2w-i)}$ times the same function of the linear persistents v_2, v_3, v_4, ... ; and that the necessary and sufficient condition for a homogeneous and isobaric function to be a pure persistent is that it have Ω for an annihilator.

Queen's College, Oxford,
March 4, 1889.

ON THE CONVERSES OF SOME THEOREMS IN ELEMENTARY GEOMETRY.

By *C. Leudesdorf.*

THE theorems in question are the well-known extensions of Ptolemy's Theorem due to Professor Casey. The converses are as follows:

(A) If a circle is such that the lengths x, y, z of the tangents to it from three points A, B, C respectively are connected by the relation $ax + by + cz = 0$ (where $BC = a$, $CA = b$, $AB = c$), then it will touch the circle circumscribing ABC.

(B) If the common tangents of four circles taken in pairs are connected by the relation

$$t_{12}t_{34} + t_{23}t_{14} + t_{31}t_{24} = 0,$$

then the circles will all touch a fifth circle.

(C) If a small circle of a sphere is such that the lengths x, y, z of the tangent arcs to it from three points A, B, C on the sphere respectively are connected by the relation

$$\sin \tfrac{1}{2}a \sin \tfrac{1}{2}x + \sin \tfrac{1}{2}b \sin \tfrac{1}{2}y + \sin \tfrac{1}{2}c \sin \tfrac{1}{2}z = 0$$

(where the arcs $BC = a$, $CA = b$, $AB = c$), then it will touch the small circle circumscribing ABC.

(*D*) If the common tangents of four small circles of a sphere taken in pairs are connected by the relation

$$\sin\tfrac{1}{2}t_{12}\sin\tfrac{1}{2}t_{34}+\sin\tfrac{1}{2}t_{23}\sin\tfrac{1}{2}t_{14}+\sin\tfrac{1}{2}t_{31}\sin\tfrac{1}{2}t_{24}=0,$$

then the four circles will all touch a fifth small circle.

These have not, so far as I know, been proved anywhere. The theorems of which they are the converses are given in several text-books, and are used at once to establish Feuerbach's theorem on the contact of the nine-point circle and the inscribed and escribed circles of a triangle, and Dr. Hart's extensions of it. But these all depend, not on the theorems themselves, but on the above converses; so that the usual method of taking the truth of these converses for granted does not seem quite satisfactory. The object of the present paper is to give simple demonstrations of them. Of course (*A*), (*B*), (*C*) are all included in (*D*), and (*A*) and (*C*) are only particular cases of (*B*) and (*D*); but it seems desirable to enunciate them separately.

§ 1. *Proof of* (*A*). Let the lengths of the tangents from three points A, B, C to a circle of given centre and radius ρ be ξ, η, ζ; and let the lengths of the tangents from the same points to any two concentric circles of radii r, r' be x, y, z and x', y', z' respectively. Then

$$\left.\begin{aligned} x^2-\xi^2&=y^2-\eta^2=z^2-\zeta^2=\rho^2-r^2=n \text{ say}\\ x'^2-\xi^2&=y'^2-\eta^2=z'^2-\zeta^2=\rho^2-r'^2=m \text{ say}\end{aligned}\right\}\dots(1),$$

so that

$$\xi^2=\frac{mx^2+nx'^2}{m+n},\quad \eta^2=\frac{my^2+ny'^2}{m+n},\quad \zeta^2=\frac{mz^2+nz'^2}{m+n}\dots(2).$$

It is evident that with the given centre there can be drawn in general two circles to touch the circle circumscribing ABC, and that these will touch it at the ends of a diameter; let the circles (r) and (r') be taken to be these. Then, if $BC=a$, $CA=b$, $AB=c$, we are given that $a\xi+b\eta+c\zeta=0$. But from the theorem to which (*A*) is converse

$$ax+by+cz=0,\ ax'+by'+cz'=0\dots\dots\dots(3);$$

therefore

$$\begin{aligned} a^2\xi^2+b^2\eta^2+2ab\xi\eta&=c^2\zeta^2,\\ a^2x^2+b^2y^2+2abxy&=c^2z^2,\\ a^2x'^2+b^2y'^2+2abx'y'&=c^2z'^2;\end{aligned}$$

from which, by (1),

$$(m+n)\,\xi\eta = mxy + nx'y'\,;$$

therefore $(mx^2+nx'^2)(my^2+ny'^2) = (mxy+nx'y')^2$,

therefore $$mn\,(xy'-x'y)^2 = 0.$$

But $xy'-x'y=0$ must be rejected; for, by (1),

$$x^2-x'^2 = y^2-y'^2 = z^2-z'^2,$$

so that, by (3), if $xy'=x'y$, then x, y, z, x', y', z' must all be zero, which is impossible. Therefore either $m=0$ or $n=0$, so that $\rho=r$ or $\rho=r'$; *i.e.*, the circle (ρ) touches the circle *ABC*.

§ 2. *Proof of* (*B*). For $t_{12}, t_{23}, t_{31}, t_{14}, t_{24}, t_{34}$ in the enunciation write $a, b, c, \xi, \eta, \zeta$ respectively, and let $\alpha, \beta, \gamma, \rho$ denote the radii of the four circles. Let two circles be drawn concentric with the circle (ρ) and having radii r, r', and let x, y, z and x', y', z' be the lengths of the common tangents of these circles and the circles $(\alpha), (\beta), (\gamma)$ respectively. Then

$$x^2-\xi^2 = y^2-\eta^2 = z^2-\zeta^2 = (\alpha\pm\rho)^2-(\alpha\pm r)^2 = n \text{ say},$$

$$x'^2-\xi^2 = y'^2-\eta^2 = z'^2-\zeta^2 = (\alpha\pm\rho)^2-(\alpha\pm r')^2 = m \text{ say};$$

therefore $$\xi^2 = \frac{mx^2+nx'^2}{m+n}, \quad \eta^2 = \&\text{c.}, \quad \zeta^2 = \&\text{c.};$$

so that if a, b, c now denote the lengths of the common tangents of the circles (β) and (γ), (γ) and (α), (α) and (β), the rest of the reasoning is precisely the same as in § 1.

§ 3. A similar method will prove (*C*) and (*D*). All that it is necessary to see is that (1) with any given pole two and only two small circles can in general be drawn to touch the circle *ABC*, and that (2) a relation similar to those of (2) holds between the squares of the sines of half the arcs ξ, x, x' of great circles which are the common tangents of any given small circle (α) and three small circles $(\rho), (r), (r')$ which have the same pole. To shew this, we have, if δ be the distance between the pole of (α) and that of the other three circles,

$$(1-\cos\xi)\ \cos\rho\cos\alpha = \cos(\rho\pm\alpha)-\cos\delta,$$

$$(1-\cos x)\ \cos r\cos\alpha = \cos(r\pm\alpha)-\cos\delta,$$

$$(1-\cos x')\cos r'\cos\alpha = \cos(r'\pm\alpha)-\cos\delta;$$

therefore

$$\begin{aligned}\sin^2\tfrac{1}{2}\xi \cos\rho \,\{\cos(r\pm\alpha)-\cos(r'\pm\alpha)\} &= \sin^2\tfrac{1}{2}x \cos r\,\{\cos(\rho\pm\alpha)-\cos(r'\pm\alpha)\}\\ &\quad+\sin^2\tfrac{1}{2}x' \cos r'\,\{\cos(r\pm\alpha)-\cos(\rho\pm\alpha)\},\end{aligned}$$

a formula which applies to the case of (D). When the circle (α) reduces to a point the formula reduces to

$$\sin^2\tfrac{1}{2}\xi \cos\rho\,(\cos r-\cos r') = \sin^2\tfrac{1}{2}x \cos r\,(\cos\rho-\cos r')+\sin^2\tfrac{1}{2}x' \cos r'\,(\cos r-\cos\rho),$$

which applies to the case of (C).

§ 4. Using the properties of the radical axis, limiting points, &c.—*which has not been done in what precedes*—another proof of (A) may be given. Suppose, if possible, that the given circle (ρ) cuts the circle (R) circumscribing ABC in two points P and P'; let δ be the distance between the centres of the circles. Then

$$\begin{aligned}\xi^2 &= 2\delta\text{ (perpendicular from } A \text{ on } PP')\\ &= 2\delta\frac{AP.AP'}{PP'}\sin PAP'\\ &= \frac{\delta}{R}AP.AP' \quad\ldots\ldots(4),\end{aligned}$$

and similarly for η^2 and ζ^2. If therefore the distances of P and P' from A, B, C be denoted by x, y, z and x', y', z' respectively, we have

$$a\sqrt{(xx')}+b\sqrt{(yy')}+c\sqrt{(zz')}=0,$$

while $\qquad ax+by+cz=0$ and $ax'+by'+cz'=0.$

But these cannot be simultaneously true. For

$$(ax+by)(ax'+by')=\{a\sqrt{(xx')}+b\sqrt{(yy')}\}^2;$$

therefore $\qquad ab\,\{\sqrt{(xy')}-\sqrt{(x'y)}\}^2=0,$

so that $\qquad \dfrac{x}{x'}=\dfrac{y}{y'}=\dfrac{z}{z'};$

but this is impossible unless P and P' coincide, and the circles touch.

Next suppose if possible that the circles (ρ) and (R) do not cut one another: then if L be one of their limiting points,

$$a.AL+b.BL+c.CL=a\xi+b\eta+c\zeta=0\ldots\ldots(5).$$

so that L must lie on the circle ABC. But this is impossible unless L lies on both circles and they touch at the point.

§ 5. A proof similar to that of the last paragraph applies also to (C). If the notation of § 2 is used, and if R be the radius of the circle circumscribing ABC, r the radius of the given circle, δ the arc of a great circle joining their poles, P and P' the points where (if possible) the circles intersect, p the perpendicular arc from A on PP', then (McClelland and Preston, *Spher. Trig.*, vol. II., p. 126)

$$\sin^2\tfrac{1}{2}x = \frac{\sin p \sin \delta}{2 \cos R \cos r} \cos \tfrac{1}{2}PP'.$$

But $\sin p \sin PP' = \sin AP \sin AP' \sin PAP'$,

and $$\tan R = \frac{\sin PP'}{\cos \frac{1}{2}AP \cos \frac{1}{2}AP' \sin PAP'};$$

therefore $$\sin^2\tfrac{1}{2}x = \frac{\sin \delta}{\sin R \cos r} \sin \tfrac{1}{2}AP \sin \tfrac{1}{2}AP',$$

a formula analogous to (4), and which may be employed in a manner similar to that in which (4) has been used.

An argument exactly similar to that of the last part of § 4 also holds here, the symbols in (5) being replaced by the sines of half the corresponding arcs of great circles.

§ 6. It may be noticed that the form of the reasoning in § 1 remains the same even if a, b, c are not the sides of the triangle ABC but are any constants whatever. If they are replaced by l, m, n, then, since the equation of the circle (r) in triangular coordinates with ABC as triangle of reference is

$$a^2YZ + b^2ZX + c^2XY = (X + Y + Z)(x^2X + y^2Y + z^2Z),$$

it is seen that if the radical axis of any circle and the circle ABC touches the conic $l^2YZ + m^2ZX + n^2XY = 0$; then the relation $lx \pm my \pm nz = 0$ holds. The converse of this can now be seen to be true. Suppose the relation just given to hold for a certain circle. If the centre of such a circle is fixed, the direction of the radical axis of it and the circle ABC is fixed; and, since only two tangents can be drawn to a conic parallel to a given direction, only two concentric circles exist, the radical axes of which with the circle ABC touch the conic in question. Therefore, by reasoning as in § 1, if any circle be such that $lx \pm my \pm nz = 0$, the radical axis of it and the circle ABC must touch the conic $l^2YZ + m^2ZX + n^2XY = 0$, a result which is an extension of (A).

ON STIRLING'S FORMULA AND OTHER INTERPOLATION FORMULÆ.

By *G. H. Stuart, M.A.*

1. THE following note is based upon two papers by Emory Mc Clintock in the *American Journal of Mathematics*, Vol. II., 1879, viz., 'An Essay on the Calculus of Enlargement,' p. 101, and 'A new general method of Interpolation,' p. 307. The theorems in those papers are expressed in a very general form, and some of the demonstrations may be much simplified when the only object is to obtain formulæ of interpolation. Mr. Clintock's notation is very expressive, and will be here used.

All the theorems given in this note are well known, but there are no very elementary proofs of most of them. In giving such proofs I have wished to make a complete *elementary* exposition of this part of the subject of Finite Differences, and therefore there is unavoidably some repetition of known processes and results.

2. There are three kinds of differences considered, viz., firstly, the ordinary kind, called by Mc Clintock "upper differences," and denoted by the symbol Δ, where

$$\Delta u_x = u_{x+1} - u_x \quad\dots\dots\dots\dots\dots\dots\dots\dots \text{(1)},$$

so that

$$\Delta = E - 1 \quad\dots\dots\dots\dots\dots\dots\dots\dots \text{(2)};$$

secondly, "central differences" denoted by the symbol Λ, where

$$\Lambda u_x = u_{x+\frac{1}{2}} - u_{x-\frac{1}{2}} \quad\dots\dots\dots\dots\dots\dots\dots\dots \text{(3)},$$

so that

$$\Lambda = E^{\frac{1}{2}} - E^{-\frac{1}{2}} \quad\dots\dots\dots\dots\dots\dots\dots\dots \text{(4)};$$

and thirdly, "mean central differences" which are the central differences of the function $\frac{1}{2}(u_{x+\frac{1}{2}} + u_{x-\frac{1}{2}})$. This quantity is denoted by the symbol Iu_x, so that

$$I = \tfrac{1}{2}(E^{\frac{1}{2}} + E^{-\frac{1}{2}}) \quad\dots\dots\dots\dots\dots\dots\dots\dots \text{(5)},$$

and the symbol for the n^{th} central difference is $\Lambda^n I$ or $I\Lambda^n$.

The central and mean central differences can be expressed as upper differences as follows

$$\Lambda = E^{\frac{1}{2}} - E^{-\frac{1}{2}} = \Delta E^{-\frac{1}{2}} \quad\dots\dots\dots\dots\dots\dots \text{(6)},$$

therefore $$\Lambda^n u_x = \Delta^n E^{-\frac{1}{2}n} u_x = \Delta^n u_{x-\frac{1}{2}n} \text{ ....(7)};$$

and $$I\Lambda^n = \tfrac{1}{2}\Delta^n E^{-\frac{1}{2}n-\frac{1}{2}}(E+1),$$

therefore $$I\Lambda^n u_x = \tfrac{1}{2}\Delta^n (u_{x-\frac{1}{2}n+\frac{1}{2}} + u_{x-\frac{1}{2}n-\frac{1}{2}}) \text{.............(8)}.$$

3. Corresponding to the three sorts of differences there are three sorts of factorials, viz., the ordinary kind, called by McClintock "upper factorials," and denoted by $x^{n)}$, where

$$x^{n)} = x(x-1)(x-2)\ldots(x-n+1) \text{...........(9)};$$

secondly, "central factorials," denoted by $x^{(n)}$, where

$$x^{(n)} = x(x+\tfrac{1}{2}n-1)(x+\tfrac{1}{2}n-2)\ldots(x-\tfrac{1}{2}n+1) \text{...(10)};$$

thirdly, "mean central factorials," denoted by $^{)n(}$, where

$$x^{)n(} = (x+\tfrac{1}{2}n-\tfrac{1}{2})(x+\tfrac{1}{2}n-\tfrac{3}{2})\ldots(x-\tfrac{1}{2}n+\tfrac{1}{2}) \text{...(11)}.$$

The relations that these factorials bear to the corresponding operations are the following:

$$\Delta x^{n)} = nx^{n-1} \text{ ....(12)},$$

$$\Lambda x^{(n)} = nx^{(n-1)} \text{......................(13)},$$

$$\Lambda x^{)n(} = nx^{)n-1(} \text{......................(14)},$$

the last being a consequence of

$$Ix^{)n(} = x^{(n)} \text{........................(15)}.$$

To complete the system, as far as our needs are concerned, we have

$$x^n = x.x \text{ to } n \text{ factors ....(16)},$$

and $$Dx^n = nx^{n-1} \text{ .... (17)},$$

where $$D = \lim. \frac{1}{h}(E^h - 1), \text{ when } h = 0 \text{...........(18)}.$$

In all these expressions n is a positive integer.

From the definitions it follows that

$$x^{1)} = x^{(1)} = x^{)1(} = x.$$

Equations (12)...(14) are used to define factorials with zero index: thus putting $n = 1$ in these equations we have

$$x^{0)} = \Delta x^{1)} = 1,$$

$$x^{(0)} = \Lambda x^{(1)} = 1,$$

$$x^{)0(} = \Lambda x^{)1(} = 1,$$

$$x^0 = Dx = 1.$$

It is evident that

$$x^{)n(} = x^{(n+1)} \div x.$$

4. Any one of these factorials can be expressed in terms of any other kind, and in particular any power of x can be expressed in terms of upper, central, or mean central factorials. Thus, for example,

$$\begin{aligned} x^5 &= x^{5)} + 10x^{4)} + 25x^{3)} + 15x^{2)} + x^{1)} \\ &= x^{(5)} + \tfrac{5}{2}x^{(3)} + \tfrac{1}{16}x^{(1)} \\ &= x^{)5(} + 5x^{)3(} + x^{)1(}. \end{aligned}$$

The general formulæ are given below, equations (39)...(43).

5. Hence, every function u_x which can be expressed in a series of positive integral powers of x can also be expressed (1) in a series of upper factorials, (2) in a series of central factorials, (3) in a series of mean central factorials. In what follows u_x is always supposed to be an "expansible" function of the above kind.

6. First, for the sake of completeness, I give a proof, slightly different from that given in Boole's *Finite Differences*, Ch. II., Art. 5., of the ordinary expansion of u_{x+h} in a series of upper factorials of x.

Assume, as is allowable by what precedes, that

$$u_{x+h} = a_0 + a_1 x^{1)} + a_2 \frac{x^{2)}}{1.2} + a_3 \frac{x^{3)}}{1.2.3} + \dots \quad \dots(19).$$

Take the "upper" difference of this with respect to x and also with respect to h; we have

$$\Delta_x u_{x+h} = a_1 + a_2 x^{1)} + a_3 \frac{x^{2)}}{1.2} + \dots \quad \dots\dots\dots(20),$$

$$\Delta_h u_{x+h} = \Delta a_0 + x^{1)} \Delta a_1 + \frac{x^{2)}}{1.2} \Delta a_2 + \dots \quad (21).$$

But $\Delta_h u_{x+h} = u_{x+h+1} - u_{x+h} = \Delta_x u_{x+h}$. Therefore the series in (20) and (21) are identical. Moreover, by putting $x = 0$ in (19) we get $a_0 = u_h$.

Therefore

$$\begin{aligned} a_1 &= \Delta a_0 = \Delta u_h, \\ a_2 &= \Delta a_1 = \Delta^2 u_h, \\ &\&\text{c.}, \quad \&\text{c.}, \end{aligned}$$

therefore $u_{x+h} = u_h + x^{1)} \Delta u_h + \dfrac{x^{2)}}{1.2} \Delta^2 u_h + \dfrac{x^{3)}}{1.2.3} \Delta^3 u_h + \dots \quad (\alpha).$

7. Next, assume

$$u_{x+h} = b_0 + b_1 x^{(1)} + b_2 \frac{x^{(2)}}{1.2} + b_3 \frac{x^{(3)}}{1.2.3} + \dots \qquad \dots(22).$$

By putting $x = 0$ (since $0^{(n)} = 0$), we get $b_0 = u_h$.

Take the "central" difference of (22) with respect to x and also with respect to h, we get

$$\Lambda_x u_{x+h} = b_1 + b_2 x^{(1)} + b_3 \frac{x^{(2)}}{1.2} + \dots \qquad \dots\dots\dots(23),$$

$$\Lambda_h u_{x+h} = \Lambda b_0 + x^{(1)} \Lambda b_1 + \frac{x^{(2)}}{1.2} \Lambda b_2 + \dots \qquad \dots(24).$$

But $\qquad \Lambda_x u_{x+h} = u_{x+h+\frac{1}{2}} - u_{x+h-\frac{1}{2}} = \Lambda_h u_{x+h}.$

Therefore the series in (23) and (24) are identical,

therefore

$$b_1 = \Lambda b_0 = \Lambda u_h,$$
$$b_2 = \Lambda b_1 = \Lambda^2 u_h,$$
&c., &c.,

therefore $u_{x+h} = u_h + x^{(1)} \Lambda u_h + \frac{x^{(2)}}{1.2} \Lambda^2 u_h + \frac{x^{(3)}}{1.2.3} \Lambda^3 u_h + \dots \qquad (\beta).$

8. Thirdly, assume

$$u_{x+h} = c_0 + c_1 x^{)1(} + c_2 \frac{x^{)2(}}{1.2} + c_3 \frac{x^{)3(}}{1.2.3} + \dots \qquad \dots(25).$$

Here we cannot find c_0 by putting $x = 0$, because $x^{)2(}$, $x^{)4(}$, ... are not zero. Perform the operation denoted by I on both sides with respect to x, then

$$Iu_{x+h} = c_0 + c_1 x^{(1)} + c_2 \frac{x^{(2)}}{1.2} + c_3 \frac{x^{(3)}}{1.2.3} + \dots \qquad \dots(26).$$

Put $x = 0$, then

$$c_0 = I_x u_{x+h} \, (x = 0)$$
$$= \tfrac{1}{2} (u_{h+\frac{1}{2}} + u_{h-\frac{1}{2}})$$
$$= Iu_h,$$

where the I now refers to h.

Take the central difference of (25) with respect to x and also with respect to h, we get

$$\Lambda_x u_{x+h} = c_1 + c_2 x^{)1(} + c_3 \frac{x^{)2(}}{1.2} + \dots \qquad \dots\dots\dots(27),$$

$$\Lambda_h u_{x+h} = \Lambda c_0 + x^{)1(} \Delta c_1 + \frac{x^{)2(}}{1.2} \Lambda c_2 + \dots \qquad \dots(28).$$

But $\Lambda_x u_{x+h} = \Lambda_h u_{x+h}$. Therefore the series in (27) and (28) are identical.

Therefore
$$c_1 = \Lambda c_0 = I\Lambda u_h,$$
$$c_2 = \Lambda c_1 = I\Lambda^2 u_h,$$
&c., &c.,

therefore
$$u_{x+h} = Iu_h + x^{)1(} I\Lambda u_h + \frac{x^{)2(}}{1.2} I\Lambda^2 u_h + \frac{x^{)3(}}{1.2.3} I\Lambda^3 u^{(} + \ldots \quad (\gamma).$$

9. Corresponding to the operation D and the quantity x^n, we have of course Taylor's Theorem
$$u_{x+h} = u_h + xDu_h + \frac{x^2}{1.2} D^2 u_h + \frac{x^3}{1.2.3} D^3 u_h + \ldots \quad (\delta).$$

10. The terms in the even places of the series (β) and the terms in the odd places of the series (γ) contain terms like $u_{h+\frac{1}{2}}$, $u_{h+\frac{3}{2}}$, &c., and these series are therefore not generally suitable for interpolation. They can be modified as follows:

Since $(-x)^{(n)} = (-)^n x^{(n)}$, we have from ($\beta$)
$$\tfrac{1}{2}(u_{x+h} + u_{h-x}) = u_h + \frac{x^{(2)}}{1.2} \Lambda^2 u_h + \frac{x^{(4)}}{1.2.3.4} \Lambda^4 u_h + \ldots \quad (29),$$
$$\tfrac{1}{2}(u_{x+h} - u_{h-x}) = x^{(1)} \Lambda u_h + \frac{x^{(3)}}{1.2.3} \Lambda^3 u_h + \ldots \quad \ldots\ldots\ldots(30),$$

and since $(-x)^{)n(} = (-)^n x^{)n(}$, we have from ($\gamma$)
$$\tfrac{1}{2}(u_{x+h} + u_{h-x}) = Iu_h + \frac{x^{)2(}}{1.2} I\Lambda^2 u_h + \frac{x^{)4(}}{1.2.3.4} I\Lambda^4 u_h + \ldots \quad (31),$$
$$\tfrac{1}{2}(u_{x+h} - u_{h-x}) = x^{)1(} I\Lambda u_h + \frac{x^{)3(}}{1.2.3} I\Lambda^3 u_h + \ldots \quad \ldots\ldots\ldots\ldots(32).$$

From (29) and (32) by addition, we get
$$u_{x+h} = u_h + x^{)1(} I\Lambda u_h + \frac{x^{(2)}}{1.2} \Lambda^2 u_h + \frac{x^{(3)}}{1.2.3} I\Lambda^3 u_h + \ldots \quad \ldots(\varepsilon).$$

This is Stirling's formula for interpolation.

If we attend to the relations expressed in (7) and (8) we can put this into the form in which it is usually given, viz.,
$$u_{x+h} = u_h + \tfrac{1}{2} x^{)1(} (\Delta u_h + \Delta u_{h-1}) + \frac{x^{(2)}}{1.2} \Delta^2 u_{h-1} + \tfrac{1}{2} \frac{x^{(3)}}{1.2.3} (\Delta^3 u_{h-1} + \Delta^3 u_{h-2})$$
$$+ \frac{x^{(4)}}{1.2.3.4} \Delta^4 u_{h-2} + \tfrac{1}{2} \frac{x^{)5(}}{1.2.3.4.5} (\Delta^5 u_{h-2} + \Delta^5 u_{h-3}) + \ldots \quad \ldots(\zeta).$$

See Boole's *Finite Differences*, Ch. III., Ex. 9, p. 59, (2nd. ed.), and De Morgan's *Differential and Integral Calculus*, p. 546.

11. If we add (30) and (31) we get

$$u_{x+h} = Iu_h + x^{(1)}\Lambda u_h + \frac{x^{)2(}}{1.2} I\Lambda^2 u_h + \frac{x^{(3)}}{1.2.3}\Lambda^3 u_h + \ldots \quad \ldots(33).$$

Change h into $h-\frac{1}{2}$ and x into $x+\frac{1}{2}$, we get

$$u_{x+h} = Iu_{h-\frac{1}{2}} + (x+\tfrac{1}{2})^{(1)}\Lambda u_{h-\frac{1}{2}} + \frac{(x+\frac{1}{2})^{)2(}}{1.2} I\Lambda^2 u_{h-\frac{1}{2}} + \ldots \quad (\eta),$$

and by the use of (7) and (8) this can be put into the form

$$u_{x+h} = \tfrac{1}{2}(u_h + u_{h-1}) + (x+\tfrac{1}{2})^{(1)}\Delta u_{h-1} + \tfrac{1}{2}\frac{(x+\frac{1}{2})^{)2(}}{1.2}\Delta^2(u_{h-1}+u_{h-2})$$

$$+ \frac{(x+\frac{1}{2})^{(3)}}{1.2.3}\Delta^3 u_{h-2} + \tfrac{1}{2}\frac{(x+\frac{1}{2})^{(4)}}{1.2.3.4}\Delta^4(u_{h-2}+u_{h-3})$$

$$+ \frac{(x+\frac{1}{2})^{(5)}}{1.2.3.4.5}\Delta^5 u_{h-3} + \ldots \quad \ldots\ldots\ldots(\theta),$$

which is the other formula given by De Morgan, *Differential and Integral Calculus*, p. 547.

12. Two other series for u_{x+h} are obtained as follows: In (31) for x write $x-\frac{1}{2}$; the result is

$$\tfrac{1}{2}(u_{x+h-\frac{1}{2}} + u_{h-x+\frac{1}{2}}) = Iu_h + \frac{(x-\frac{1}{2})^{)2(}}{1.2} I\Lambda^2 u_h + \frac{(x-\frac{1}{2})^{)4(}}{1.2.3.4} I\Lambda^4 u_h + \ldots \quad (34).$$

Change h into $h+\frac{1}{2}$ in (32); that is, operate with $E^{\frac{1}{2}}$ with respect to h. The result is

$$\tfrac{1}{2}(u_{x+h+\frac{1}{2}} - u_{h-x+\frac{1}{2}}) = x^{)1(} I\Lambda E^{\frac{1}{2}} u_h + \frac{x^{)3(}}{1.2.3} I\Lambda^3 E^{\frac{1}{2}} u_h + \ldots \quad \ldots(35).$$

Add (34) and (35); we get

$\frac{1}{2}(u_{x+h+\frac{1}{2}} + u_{x+h-\frac{1}{2}})$ or Iu_{x+h}

$$= Iu_h + x^{)1(}\Lambda E^{\frac{1}{2}} Iu_h + \frac{(x-\frac{1}{2})^{)2(}}{1.2}\Lambda^2 Iu_h + \frac{x^{)3(}}{1.2.3}\Lambda^3 E^{\frac{1}{2}} Iu_h + \ldots \quad \ldots(36).$$

For Iu_h write v_h; then $Iu_{x+h} = v_{x+h}$, and (36) becomes

$$v_{x+h} = v_h + x^{)1(}\Lambda E^{\frac{1}{2}} v_h + \frac{(x-\frac{1}{2})^{)2(}}{1.2}\Lambda^2 u_h + \frac{x^{)3(}}{1.2.3}\Lambda^3 E^{\frac{1}{2}} u_h + \dots \quad \dots(\kappa).$$

In (31) for x write $x + \frac{1}{2}$; the result is

$$\tfrac{1}{2}(u_{x+h+\frac{1}{2}} + u_{h-x-\frac{1}{2}}) = Iu_h + \frac{(x+\frac{1}{2})^{)2(}}{1.2} I\Lambda^2 u_h + \frac{(x+\frac{1}{2})^{)4(}}{1.2.3.4} I\Lambda^4 u_h + \dots \quad \dots\dots\dots\dots(37).$$

Change h into $h - \frac{1}{2}$ in (32); that is, operate with $E^{-\frac{1}{2}}$ with respect to h. The result is

$$\tfrac{1}{2}(u_{x+h-\frac{1}{2}} - u_{h-x-\frac{1}{2}}) = x^{)1(} I\Lambda E^{-\frac{1}{2}} u_h + \frac{x^{)3(}}{1.2.3} I\Lambda^3 E^{-\frac{1}{2}} u_h + \dots \quad \dots(38).$$

Add (37) and (38), and write v_h for Iu_h on the right, and therefore v_{x+h} for Iu_{x+h} or $\frac{1}{2}(u_{x+h+\frac{1}{2}} + u_{x-h-\frac{1}{2}})$ on the left. We thus obtain

$$v_{x+h} = v_h + x^{)1(}\Lambda E^{-\frac{1}{2}} v_h + \frac{(x+\frac{1}{2})^{)2(}}{1.2}\Lambda^2 v_h + \frac{x^{)3(}}{1.2.3}\Lambda^3 E^{-\frac{1}{2}} v_h + \dots \quad \dots(\lambda).$$

By means of the relation (7), formulæ (κ) and (λ) can be written as follows, writing also u for v,

$$u_{x+h} = u_h + x^{)1(}\Delta u_h + \frac{(x-\frac{1}{2})^{)2(}}{1.2}\Delta^2 u_{h-1} + \frac{x^{)3(}}{1.2.3}\Delta^3 u_{h-1} + \frac{(x-\frac{1}{2})^{)4(}}{1.2.3.4}\Delta^4 u_{h-2} + \dots \quad \dots(\mu),$$

$$u_{x+h} = u_h + x^{)1(}\Delta u_{h-1} + \frac{(x+\frac{1}{2})^{)2(}}{1.2}\Delta^2 u_{h-1} + \frac{x^{)3(}}{1.2.3}\Delta^3 u_{h-2} + \frac{(x+\frac{1}{2})^{)4(}}{1.2.3.4}\Delta^4 u_{h-2} + \dots \quad \dots(\nu),$$

which are identical with the formulæ given in Boole's *Finite Differences*, Ch. III., Ex. 7, p. 59. The first of them is in effect given in De Morgan's *Differential and Integral Calculus*, at the top of page 546, where, however, there is a misprint; the value given for Q_x should have 1.2.3 instead of 1.2. in the denominator.

13. Series (ε) contains alternately "central" and "mean central" differences; but this series of differences is very commonly spoken of as "central" differences. They are

most easily calculated in the form in which they are given in (ζ), as shown in the following scheme, where h is taken to be zero:

$$\begin{array}{ccccccc}
u_3 & & & & & & \\
 & a_2 & & & & & \\
u_2 & & b_1 & & & & \\
 & a_1 & & c_0 & & & \\
u_1 & & b_0 & & d_{-1} & & \\
 & a_0 & & c_{-1} & & e_{-2} & \\
u_0 & (a) & b_{-1} & (c) & d_{-2} & (e) & f_{-3} \\
 & a_{-1} & & c_{-2} & & e_{-3} & \\
u_{-1} & & b_{-2} & & d_{-3} & & \\
 & a_{-2} & & c_{-3} & & & \\
u_{-2} & & b_{-3} & & & & \\
 & a_{-3} & & & & & \\
u_{-3} & & & & & &
\end{array}$$

Here the left-hand column consists of the given values of the function u_x, as far as they are to be used, arranged around the central value u_0: the second column consists of the first "upper" differences of u; thus $a_0 = \Delta u_0 = u_1 - u_0$, &c.: the third column consists of the second upper differences of u, or the first upper differences of a; thus $b_1 = \Delta a_1 = a_2 - a_1$, &c.: and so on.

The central differences used in (ζ) are the terms in the same horizontal line as u_0; where there is no term in this line (as for instance in the "a" column) one is obtained by taking the arithmetical mean between the term immediately above and that immediately below. These insinuated terms are enclosed in brackets (a), (c), (e), &c., where $(a) = \frac{1}{2}(a_0 + a_{-1})$, $(c) = \frac{1}{2}(c_{-1} + c_{-2})$, $(e) = \frac{1}{2}(e_{-2} + e_{-3})$, &c.

In the same way, the differences used in (θ) are found from the same scheme by taking all the terms in the same horizontal line as a_{-1} and supplying the missing terms in any column by taking the mean of the adjacent terms. These will be (u), a_{-1}, (b), c_{-2}, (d), e_{-3}, &c.: where $(u) = \frac{1}{2}(u_0 + u_{-1})$, $(b) = \frac{1}{2}(b_{-1} + b_{-2})$, $(d) = \frac{1}{2}(d_{-2} + d_{-3})$, &c.

The differences in (μ) and (ν) are found from the same scheme by following the zigzag lines u_0, a_0, b_{-1}, c_{-1}, d_{-2}, e_{-2}, f_{-3} and u_0, a_{-1}, b_{-1}, c_{-2}, d_{-2}, e_{-3}, f_{-3} respectively; so that they also may be regarded as "quasi-central" differences.

The differences in (α), h being zero, follow the ascending line u_0, a_0, b_0, c_0, &c.

The differences in (β) and (γ) require a knowledge of $u_{\frac{1}{2}}$, $u_{-\frac{1}{2}}$, $u_{\frac{3}{2}}$, $u_{-\frac{3}{2}}$, &c., and are not fully expressed in this scheme.

14. The reasons for using central differences instead of upper differences in certain cases of interpolation are given by De Morgan, *Differential and Integral Calculus*, pp. 544, 545. If we are seeking an approximation to the value of u_z, where z lies between h and $h+1$ ($z=h+x$ where x is a positive proper fraction), and if we agree to neglect fourth differences, then, to obtain the first three orders of differences, we may use the given values u_h, u_{h+1}, u_{h+2}, u_{h+3}, and substitute in (α). But it is clearly better to use values of u more equally distributed round u_z as centre; *i.e.*, we shall get a better approximation if we use the values u_{h-1}, u_h, u_{h+1}, u_{h+2}, and this amounts to using one of the central difference formulæ (viz., (θ), putting $h+1$ for h, $x-1$ for x in that formula). Or, if fifth differences are neglected, it is better to obtain the four orders of differences which are to be used from the values u_{h-2}, u_{h-1}, u_h, u_{h+1}, u_{h+2}; and this amounts to using Stirling's formula (ζ). But if the values preceding u_h are not known, then these formulæ cannot be used, and the value of u_z must be found from (α).

Of course all the formulæ (α)...(ν) are really the same, and give the same result when all the terms are used; it is only when an approximate value is sought that a choice exists among them.

15. In (α), (β), (γ) let $u_x = x^m$, and put $h=0$; then

$$x^m = x\Delta 0^m + \frac{x^{2)}}{1.2}\Delta^2 0^m + \ldots + \frac{x^{m)}}{m!}\Delta^m 0^m \ldots\ldots.(39),$$

$$x^m = x\Lambda 0^m + \frac{x^{(3)}}{3!}\Lambda^3 0^m + \ldots + \frac{x^{(m)}}{m!}\Lambda^m 0^m \ldots\ldots.(40),$$

if m is odd; and

$$x^m = \frac{x^{(2)}}{2!}\Lambda^2 0^m + \frac{x^{(4)}}{4!}\Lambda^4 0^m + \ldots + \frac{x^{(m)}}{m!}\Lambda^m 0^m \ldots\ldots(41),$$

if m is even.

$$x^m = xI\Lambda 0^m + \frac{x^{)3(}}{3!}I\Lambda^3 0^m + \ldots + \frac{x^{)m(}}{m!}I\Lambda^m 0^m \ldots\ldots(42),$$

if m is odd; and

$$x^m = (\tfrac{1}{2})^m + \frac{x^{)2(}}{2!}I\Lambda^2 0^m + \ldots + \frac{x^{)m(}}{m!}I\Lambda^m 0^m \ldots\ldots(43),$$

if m is even.

These are the general formulæ before alluded to; and it is clear that in the same way any factorial of one kind may be expressed as a series of factorials of any other kind. This theorem is given in its most general form by Mc Clintock, *l.c.*, p. 143, eq. (297). And see also *Math. Mess.*, (First Series), vol. v., 1871, p. 105, where Prof. Cayley shows how to develope a product such as $(h-a)(h-b)(h-c)(h-d)(h-e)$ in a series of products 1, $(h-\alpha)$, $(h-\alpha)(h-\beta)$, $(h-\alpha)(h-\beta)(h-\gamma)$, &c., by the method of varied multiplication.

16. If in (α) we interchange x and h, we get

$$u_{x+h} = u_x + h\Delta u_x + \frac{h^{(2)}}{2!}\Delta^2 u_x + \dots,$$

or

$$u_{x+h} - u_x = h\Delta u_x + \frac{h(h-1)}{1.2}\Delta^2 u_x + \dots.$$

If we now divide by h and make $h=0$, we get the ordinary series for a differential coefficient in terms of a series of differences, viz.:

$$Du_x = \Delta u_x - \tfrac{1}{2}\Delta^2 u_x + \tfrac{1}{3}\Delta^3 u_x - \dots \quad \dots\dots\dots(44).$$

17. If we proceed in the same way with the series (β), we get

$$Du_x = \Lambda u_x + \frac{0^{)2(}}{3!}\Lambda^3 u_x + \frac{0^{)4(}}{5!}\Lambda^5 u_x + \dots \quad \dots\dots(45),$$

because $h^{(n+1)} = h.h^{)n(}$ and so lim. $(h=0)$ of $h^{(n+1)} \div h$ is $0^{)n(}$.

Substituting the values of $0^{)2(}$, $0^{)4(}$, &c., we get

$$Du_x = \Lambda u_x - \tfrac{1}{24}\Lambda^3 u_x + \tfrac{3}{640}\Lambda^5 u_x - \tfrac{5}{7168}\Lambda^7 u_x + \dots \quad \dots(46).$$

18. Proceeding in the same way with the series (ε), we get

$$Du_x = I\Lambda u_x + \frac{0^{)3(}}{0\,(3!)}I\Lambda^3 u_x + \frac{0^{)5(}}{0\,(5!)}I\Lambda^5 u_x + \dots \quad \dots(47),$$

where $\dfrac{0^{)m(}}{0}$ means the limit of $\dfrac{h^{)m(}}{h}$ when h is zero: this limit is finite when m is odd, because in that case $h^{)m(}$ contains the factor h.

Substituting the values of $\dfrac{0^{)3(}}{0}$, &c., we get

$$Du_x = I\Lambda u_x - \frac{1^2}{3!}I\Lambda^3 u_x + \frac{(1.2)^2}{5!}I\Lambda^5 u_x - \frac{(1.2.3)^2}{7!}I\Lambda^7 u_x + \dots \quad \dots(48),$$

$$= I\Lambda u_x - \tfrac{1}{6}I\Lambda^3 u_x + \tfrac{1}{30}I\Lambda^5 u_x - \tfrac{1}{140}I\Lambda^7 u_x + \dots \quad \dots\dots\dots(49),$$

which is a very useful theorem. If third differences can be neglected, we get, as an approximate value of Du_x,

$$Du_x = \tfrac{1}{2}(u_{x+1} - u_{x-1});$$

and if third differences are retained, and fifth differences are neglected, we get

$$Du_x = \tfrac{2}{3}(u_{x+1} - u_{x-1}) - \tfrac{1}{12}(u_{x+2} - u_{x-2}),$$

and so on.

Equations (46) and (49) are Mc Clintock's equations (254) and (256), *l.c.*, p. 138.

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NOTE ON THE SUMS OF TWO SERIES.

By Prof. *Cayley*.

I CONSIDER the two series

$$S = \frac{1}{1+e^{\pi\alpha}} + \frac{1}{3(1+e^{3\pi\alpha})} + \frac{1}{5(1+e^{5\pi\alpha})} + \ldots,$$

and
$$S_1 = \frac{1}{2+\pi\alpha} + \frac{1}{3(2+3\pi\alpha)} + \frac{1}{5(2+5\pi\alpha)} + \ldots,$$

where α is real, positive, and indefinitely small; these would at first sight appear to be equal to each other, but this is not in fact the case.

Taking first the series S_1, putting therein $\pi\alpha = 2x$, this is

$$2S_1 = \frac{1}{1+x} + \frac{1}{3(1+3x)} + \frac{1}{5(1+5x)} + \ldots.$$

Now we have (Legendre, *Théorie des Fonctions Elliptiques*, t. II., p. 438)

$$\frac{y}{1+y} + \frac{y}{2(2+y)} + \frac{y}{3(3+y)} + \ldots = C + \frac{d}{dy}\log\Gamma(y+1),$$

where C is Euler's constant, $= \cdot 577\ldots$; and if y be real, positive, and very large, then

$$\Gamma(y+1) = \sqrt{(2\pi)}\, y^{y+\frac{1}{2}} e^{-y+\frac{1}{12y}+\ldots};$$

whence, differentiating the logarithm and neglecting the terms which contain negative powers of y, then the value is $= C + \log y$; hence, writing $y = \frac{1}{x}$, we obtain

$$\frac{1}{1+x} + \frac{1}{2(1+2x)} + \frac{1}{3(1+3x)} + \ldots = C - \log x.$$

Writing herein $2x$ for x, and dividing by 2, we have

$$\frac{1}{2(1+2x)}+\frac{1}{4(1+4x)}+\ldots=\tfrac{1}{2}C-\tfrac{1}{2}\log 2x, =\tfrac{1}{2}(C-\log 2)-\tfrac{1}{2}\log x;$$

or, subtracting,

$$\frac{1}{1+x}+\frac{1}{3(1+3x)}+\frac{1}{5(1+5x)}+\ldots=\tfrac{1}{2}(C+\log 2)-\tfrac{1}{2}\log x.$$

Hence, writing for x its value, $=\tfrac{1}{2}\pi\alpha$, we have

$$S_1=\tfrac{1}{4}(C+\log 2)-\tfrac{1}{4}\log\tfrac{1}{2}\pi\alpha, \quad =\tfrac{1}{4}(C+2\log 2-\log\pi)-\tfrac{1}{4}\log\alpha.$$

For the series S we have (*Fundamenta Nova*, p. 103) the formula

$$\tfrac{1}{4}\log\frac{2K}{\pi}=\frac{1}{1+q^{-1}}+\frac{1}{3(1+q^{-3})}+\frac{1}{5(1+q^{-5})}+\ldots;$$

or, putting herein $\alpha=\frac{K'}{K}$, then $q=e^{-\frac{\pi K'}{K}}=e^{-\pi\alpha}$, and thence

$$S=\frac{1}{1+e^{\pi\alpha}}+\frac{1}{2(1+e^{2\pi\alpha})}+\frac{1}{3(1+e^{3\pi\alpha})}+\ldots, \quad =\tfrac{1}{4}\log\frac{2K}{\pi};$$

we have $q=e^{-\pi\alpha}$, which is real, positive, and less than but indefinitely near to 1; hence also k is real, positive, and less than but indefinitely near to 1, say the value is $=1-\beta$; thence $k'=\sqrt{(2\beta)}$, and $K=\log\frac{4}{k'}, =\log\frac{2\sqrt{2}}{\sqrt{\beta}}$; also $K'=\tfrac{1}{2}\pi$, and therefore $\alpha=\frac{K'}{K}=\tfrac{1}{2}\pi\div\log\frac{2\sqrt{2}}{\sqrt{\beta}}$, whence $\log\frac{2\sqrt{2}}{\sqrt{\beta}}=\frac{\pi}{2\alpha}$, which is the relation between α and β; and we thus have $\frac{2K}{\pi}=\frac{2}{\pi}\log\frac{2\sqrt{2}}{\sqrt{\beta}}, =\frac{1}{\alpha}$; and consequently $S=\tfrac{1}{4}\log\frac{2K}{\pi}$, $=-\tfrac{1}{4}\log\alpha$. The two values thus are

$$S=-\tfrac{1}{4}\log\alpha, \quad S_1=\tfrac{1}{4}(C+2\log 2-\log\pi)-\tfrac{1}{4}\log\alpha,$$

each depending on $\log\alpha$, and having for this term the same coefficient $-\tfrac{1}{4}$; but there is in S_1 a constant term $\tfrac{1}{4}(C+2\log 2-\log\pi)$, where C is the constant ·577... .

It is easy to see why the series S is not reducible to S_1; however small α may be in the general term $\frac{1}{n(1+e^{n\pi\alpha})}$, taking n sufficiently large, not only $n\pi\alpha$ is not indefinitely small, but it in fact becomes indefinitely large; the general term of the first series thus approximates to $\frac{1}{ne^{n\pi\alpha}}$, or the terms diminish somewhat more rapidly than in a geometric series with the ratio $e^{-\pi\alpha}$ (a small positive value less than but very near to 1), whereas in the second series the general term approximates to $\frac{1}{n^2\pi\alpha}$, or the convergence is ultimately that of the series $\frac{1}{n^2}+\frac{1}{(n+1)^2}+\dots$.

NOTE ON ENERGY IN AN ELASTIC SOLID.

By *Karl Pearson.*

1. LET us consider the portion of an elastic solid bounded by the closed surface S, and further assume no body-force or source of disturbance to act upon or in the portion of the elastic solid enclosed by S. Then the indestructibility of energy forces us to the conclusion that under these circumstances any change of energy within S must be due to energy which has crossed the surface of S. Suppose that in the case with which we are to deal there can be no transfer of energy into other forms than those of vibratory energy of the body or medium and its strain-energy. For example, this would be the case in the ether at parts unoccupied by matter, supposing it to be an elastic medium capable only of radiating and not conducting or absorbing. I propose to consider *how* the energy is transferred across the surface S.

The surface S may be considered fixed in space for all the interval dealt with, or it may be taken as a surface in the unstrained elastic medium, which itself receives strain. In the latter case we will denote it by S', in the former by S.

2. Let the three stretches parallel to three rectangular directions be s_x, s, s_z and the corresponding slides be σ_{yz}, σ_{zx}, σ_{xy}; further let the three twists, or

$$\frac{1}{2}\left(\frac{dw}{dy}-\frac{dv}{dz}\right),\quad \frac{1}{2}\left(\frac{du}{dz}-\frac{dw}{dx}\right),\quad \frac{1}{2}\left(\frac{dv}{dx}-\frac{du}{dy}\right),$$

where u, v, w are the three shifts of the points x, y, z of the unstrained medium, be represented by τ_{yz}, τ_{zx}, τ_{xy}.

Then, if ρ be the density, λ and μ the dilatation and slide coefficients we have on Green's theory for an isotropic medium:

Strain-energy per unit-volume at x, y, z,

$$=\tfrac{1}{2}\{\lambda\theta^2+2\mu(s_x^2+s_y^2+s_z^2)+\mu(\sigma_{yz}^2+\sigma_{zx}^2+\sigma_{xy}^2)\},$$

θ being the dilatation.

Representing time-fluxions by superposed dots, the kinetic energy per unit-volume will be $\tfrac{1}{2}\rho(\dot{u}^2+\dot{v}^2+\dot{w}^2)$. Thus, if E be the total energy in S, the surface supposed to remain fixed in space, it will be given by

$$2E=\iiint\{\rho(\dot{u}^2+\dot{v}^2+\dot{w}^2)+\lambda\theta^2+2\mu(s_x^2+s_y^2+s_z^2)$$
$$+\mu(\sigma_{yz}^2+\sigma_{zx}^2+\sigma_{xy}^2)\}\,dx\,dy\,dz\text{ ....(i)},$$

where the three-fold integrals denote a volume summation throughout S.

Taking the time change of energy we find

$$\frac{dE}{dt}=\iiint\{\rho(\dot{u}\ddot{u}+\dot{v}\ddot{v}+\dot{w}\ddot{w})+\lambda\theta\dot{\theta}+2\mu(s_x\dot{s}_x+s_y\dot{s}_y+s_z\dot{s}_z)$$
$$+\mu(\sigma_{yz}\dot{\sigma}_{yz}+\sigma_{zx}\dot{\sigma}_{zx}+\sigma_{xy}\dot{\sigma}_{xy})\}\,dx\,dy\,dz\text{ ....(ii)}.$$

Now the shift equations of elasticity in the body of the medium are of the type

$$\rho\ddot{u}=(\lambda+\mu)\frac{d\theta}{dx}+\mu\nabla^2u\text{ ....(iii)},$$

where ∇^2 represents the Laplacian.

Substituting types of this form for $\rho\ddot{u}$, $\rho\ddot{v}$, ρ^{w} and integrating by parts we find

$$\frac{dE}{dt}=\lambda\iint\theta(l\dot{u}+m\dot{v}+n\dot{w})\,dS$$
$$+\mu\iint\dot{u}\left(2l\frac{du}{dx}+m\sigma_{xy}+n\sigma_{zx}\right)dS$$
$$+\mu\iint\dot{v}\left(2m\frac{dv}{dy}+n\sigma_{yz}+l\sigma_{xy}\right)dS$$
$$+\mu\iint\dot{w}\left(2n\frac{dw}{dz}+l\sigma_{zx}+m\sigma_{yz}\right)dS\text{ ....(iv)},$$

where l, m, n are the direction-cosines of the normal to the element dS of the surface S and the two-fold integrations are surface-summations over S.

The last three lines may be rearranged from the identities of type

$$2l\frac{du}{dx}+m\sigma_{xy}+n\sigma_{zx}=2\left(l\frac{du}{dx}+m\frac{du}{dy}+n\frac{du}{dz}\right)+2\left(m\tau_{xy}-n\tau_{zx}\right).$$

Thus after re-arranging we may re-write the value for the change in E

$$\frac{dE}{dt}=\lambda\iint\theta\,(l\dot{u}+m\dot{v}+n\dot{w})\,dS$$

$$+\,2\mu\iint\{\dot{u}\,(m\tau_{xy}-n\tau_{zx})+\dot{v}\,(n\tau_{yz}-l\tau_{xy})+\dot{w}\,(l\tau_{zx}-m\tau_{yz})\}\,dS$$

$$+\,2\mu\iint\left(\dot{u}\frac{du}{dn}+\dot{v}\frac{dv}{dn}+\dot{w}\frac{dw}{dn}\right)dS\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(v)},$$

where dn is an element of the normal measured inwards.

Thus the flow of energy into S consists of three different kinds represented by the three several lines on the right of equation (v). Let us term these components F_1, F_2, F_3, and investigate a little more closely their nature.

3. *On the nature of the energy-flux F_1.*

Let V = the resultant shift-velocity and let l', m', n' be its direction cosines; let χ be the angle between the resultant shift-velocity and the normal to dS, then

$$\cos\chi=ll'+mm'+nn',$$

and we find

$$F_1=\lambda\iint\theta\,V\cos\chi\,dS,$$

or, if U be the shift-velocity normal to dS, we see that F_1 is the flux of $\lambda\theta U$ across the surface. Thus the first kind of energy is transferred in the direction of motion, and across unit area per unit time its magnitude $=\lambda\theta U$, where U is the velocity component perpendicular to the area. We note that this factor of the flow of energy (i) vanishes if there is no dilatation, (ii) is negligible if λ is very small as compared with μ.

The stress-strain equations of the type

$$\widehat{xx}=\lambda\theta+2\mu s_x,$$

where $\widehat{xx}$ denotes the traction on an elementary area perpendicular to x lead to

$$\widehat{xx}+\widehat{yy}+\widehat{zz}=(3\lambda+2\mu)\,\theta$$
$$=-3p,\text{ say},$$

where $p = -\frac{\widehat{xx} + \widehat{yy} + \widehat{zz}}{3}$ may be spoken of as the mean pressure 'at the point.' Thus this flux per unit area may be represented by

$$-\frac{3\lambda}{3\lambda + 2\mu} p\dot{U}.$$

In the case where λ is very large as compared with μ, this equals

$$-p\dot{U},$$

or is identical in form with the flow of energy across the surface due to ordinary fluid pressure.

Extending the conception of 'fluid pressure' we may speak of

$$F_1 = -\frac{3\lambda}{3\lambda + 2\mu} \iint p\dot{U} dS \ldots\ldots\ldots\ldots\ldots\ldots \text{(vi)},$$

as the "flow of pressural energy." Thus we see that pressural energy is always transferred in the direction of motion.

4. *On the nature of the energy flux* F_2.

Let ω be the resultant twist and let it make angles whose direction cosines are l'', m'', n'' with the axes, so that $\tau_{yz} = l''\omega$, $\tau_{zx} = m''\omega$, and $\tau_{xy} = n''\omega$.

We easily find

$$F_2 = 2\mu \iint V\omega \{l'(mn'' - m''n) + m'(nl'' - ln'') + n'(lm'' - ml'')\} dS.$$

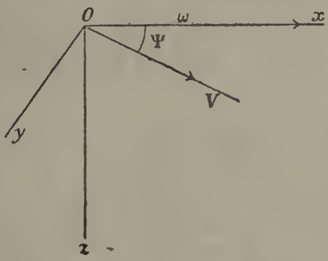
Now the factor in curled brackets vanishes when l, m, n are equal to either l', m', n' or to l'', m'', n''. Hence it follows that in this case *the transfer of energy is in a direction perpendicular alike to the resultant shift-velocity and to the resultant twist.*

Let ψ be the angle between the resultant velocity and the resultant twist, and ϕ be the angle between the direction in which the energy is transferred and the normal to dS, then we may write

$$F_2 = 2\mu \iint V\omega \sin\psi \cos\phi \, dS \ldots\ldots\ldots\ldots \text{(vii)}.$$

Thus the amount of energy propagated per unit element of area is $2\mu\omega V \sin\psi$, or is the product of the slide-modulus into four times the area of the triangle on the twist and shift-speed rotors as sides. For the same scalar values of the twists and shift-speeds this energy flux is zero if the directions of twist and shift-speed coincide, and it is a maximum when these quantities are at right angles.

In order to see exactly the direction of propagation, let us suppose a spectator to stand with his back to the twist axis and face the resultant velocity so that its direction is from his feet towards his head, and let Ψ be the angle between the positive senses of ω and V, the former positive sense being that indicated by the point of a right-handed screw, which would give the same direction of rotation. Then take the twist axis for axis of x; let the axis of z be perpendicular to the plane of ω and V and to the spectator's left, while the axis of y is in the plane of ω, V, and in a sense indicated by a right-handed screw round Ox, turning y towards z. Then, if the normal be taken to coincide with Oz, we shall have,



$$l = m = 0, \quad n = 1, \quad l'' = 1, \quad m'' = n'' = 0,$$
$$l' = \cos\Psi, \quad m' = \sin\Psi, \quad n' = 0.$$

Hence the element

$$2\mu V\omega \{l'(mn'' - m''n) + m'(nl'' - n''l) + n'(lm'' - ml'')\}$$

becomes equal to $2\mu V\omega \sin\Psi$

in the direction of the normal or Oz. That is, *the flow is from the spectator's right- to his left-hand.*

4. It will be seen that F_2 (like F_3 to be considered later) depends only on the slide-modulus μ of the medium.

It is convenient to speak of F_2 as the energy-flux due to twist. If light and electricity are to be explained by the vibrations of an elastic medium, the energy-flux due to twist obeys the same laws as Professor Poynting has shewn to hold for the flow of energy in the electro-magnetic field. Proper assumptions must of course be made with regard to the identity of the elastic and electro-magnetic quantities. These I shall consider in the next article.

5. We may write equation (iii) in the form

$$\frac{d\dot{u}}{dt} = \frac{\mu}{\rho}\nabla^2 u + \frac{\lambda+\mu}{\rho}\frac{d\theta}{dx};$$

whence, remembering that

$$\nabla^2 u = -\frac{d\tau_{xy}}{dy} + \frac{d\tau_{zx}}{dz} + \tfrac{1}{2}\frac{d\theta}{dx},$$

we find

$$\frac{d\dot{u}}{dt} = \frac{2\mu}{\rho}\left(\frac{d\tau_{zx}}{dz} - \frac{d\tau_{xy}}{dy}\right) + \frac{\lambda+2\mu}{\rho}\frac{d\theta}{dx}.$$

Further
$$\tau_{yz} = \tfrac{1}{2}\left(\frac{dw}{dy} - \frac{dv}{dz}\right),$$

or,
$$\frac{d\tau_{yz}}{dt} = \tfrac{1}{2}\left(\frac{d\dot{w}}{dy} - \frac{d\dot{v}}{dz}\right).$$

Now let us take

$$\begin{aligned} A\tau_{yz} &= P, & B\dot{u} &= \alpha, \\ A\tau_{zx} &= Q, & B\dot{v} &= \beta, \\ A\tau_{xy} &= R, & B\dot{w} &= \gamma, \end{aligned}$$

where A and B are certain constants, then we have

$$\left.\begin{aligned} \frac{dP}{dt} &= k_1\left(\frac{d\gamma}{dy} - \frac{d\beta}{dz}\right), \\ \frac{dQ}{dt} &= k_1\left(\frac{d\alpha}{dz} - \frac{d\gamma}{dx}\right), \\ \frac{dR}{dt} &= k_1\left(\frac{d\beta}{dx} - \frac{d\alpha}{dy}\right), \\ \frac{d\alpha}{dt} &= k_2\left(\frac{dQ}{dz} - \frac{dR}{dy}\right) + \frac{d\psi}{dx}, \\ \frac{d\beta}{dt} &= k_2\left(\frac{dR}{dx} - \frac{dP}{dz}\right) + \frac{d\psi}{dy}, \\ \frac{d\gamma}{dt} &= k_2\left(\frac{dP}{dy} - \frac{dQ}{dx}\right) + \frac{d\psi}{dz}, \end{aligned}\right\} \quad\ldots\ldots\ldots\ldots (viii),$$

where $\psi = B\dfrac{\lambda+2\mu}{\rho}\theta$, $A = 2Bk_1$, and $\sqrt{(k_1k_2)} = \sqrt{\left(\dfrac{\mu}{\rho}\right)}$, the velocity of propagation of twist waves.

If we suppose $\psi=0$, these equations agree with those obtained from Maxwell's theory* for the propagation of the electro-magnetic disturbance through the parts of an etherial field where there is no *source* of disturbance. They are identical with those used by Hertz in his recent paper in *Wiedemann's Annalen* (pp. 1, *et seq*., January, 1889) to explain the interference and refraction of electric rays. Thus we see that Poynting's Theorem becomes a special case of the general law which holds for the flow of the 'twist energy' F_2. At the same time that theorem does not account for the whole motion of the energy in an elastic medium. Probably in the ether we may take either λ or θ zero, so that we have not to deal with any flow of 'pressural energy,' it still remains however to deal with the energy flow of type F_3.

Before doing so, one or two remarks may be made with regard to equations (viii). These equations are perfectly reciprocal with regard to P, Q, R and α, β, γ, on the supposition that ψ is zero or negligible. Hence it seems to be indifferent, so far as the mere results of interference are concerned, whether we identify P, Q, R with the components of electro-motive force, and α, β, γ with those of magnetic force, or *vice versâ*. In the former case 'electric displacement' will correspond to 'elastic twist' in the medium, and magnetic induction to shift-velocity. In the latter it will be 'electric displacement' which will correspond to a velocity and magnetic induction to the twist. In neither case will 'electric displacement' correspond to a real elastic displacement. In the former the electric current corresponds to a rate of change of twist; in the latter to a shift-acceleration.

6. *On the nature of the energy-flux* F_3.

We have seen that this is given by

$$F_3=2\mu\iint\left(\dot{u}\,\frac{du}{dn}+\dot{v}\,\frac{dv}{dn}+\dot{w}\,\frac{dw}{dn}\right)dS.$$

The following identity may be proved at once by equating the coefficients of $\dot{u}$, $\dot{v}$, $\dot{w}$ on either side each to each, when they will be found identical:

---

* *Electricity and Magnetism*, vol. II., easily deduced from equations (A) of Art. 591, (E) of Art. 616, and (15) to (17) of Art. 790.

$$
\begin{aligned}
\dot{u}\frac{du}{dn} + \dot{v}\frac{dv}{dn} + \dot{w}\frac{dw}{dn} \\
= \quad & \left(\dot{u}\frac{du}{dx} + \dot{v}\frac{du}{dy} + \dot{w}\frac{du}{dz}\right) l \\
& + \left(\dot{u}\frac{dv}{dx} + \dot{v}\frac{dv}{dy} + \dot{w}\frac{dv}{dz}\right) m \\
& + \left(\dot{u}\frac{dw}{dx} + \dot{v}\frac{dw}{dy} + \dot{w}\frac{dw}{dz}\right) n \\
& - 2\left\{\dot{u}\left(m\tau_{xy} - n\tau_{zx}\right) + \dot{v}\left(n\tau_{yz} - l\tau_{xy}\right) + \dot{w}\left(l\tau_{zx} - m\tau_{yz}\right)\right\} \\
& \qquad \ldots\ldots\ldots\ldots\ldots\ldots\text{(ix)}.
\end{aligned}
$$

Let us replace $\dot{u}$, $\dot{v}$, $\dot{w}$ by Vl', Vm', Vn', and read

$$X = l'\frac{du}{dx} + m'\frac{du}{dy} + n'\frac{du}{dz},$$

with corresponding expressions for Y and Z. Then the first three lines on the right-hand side may be read

$$V(lX + mY + nZ).$$

Multiply both sides of (ix) by 2μ and integrate over S, we have

$$
\begin{aligned}
F_3 &= 2\mu \iint V(lX + mY + nZ)\, dS - 2F_2 \\
&= F_3' - 2F_2, \quad \text{say.}
\end{aligned}
$$

Thus we find

$$
\left.\begin{aligned}
\frac{dE}{dt} &= F_1 + F_2 + F_3 \\
&= F_1 - F_2 + F_3'
\end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ldots\text{(x)}.
$$

If then we can interpret F_3', we shall really have explained physically the whole flow of energy, which can be expressed by either of the forms (x). All that is necessary is to suppose that *the energy flux due to twist is from left- to right-hand* (and no longer from right- to left-hand) of a spectator with his back along the twist axis and facing the shift-velocity.

We shall see that X, Y, Z are the components of a vector f_V, whose direction is l''', m''', n'''. Thus we may write, if $\cos\chi' = ll''' + mm''' + nn'''$, or χ' be the angle between the normal to dS and f_V,

$$F_3' = 2\mu \iint V f_V \cos\chi' dS \ldots\ldots\ldots\ldots\ldots\text{(xi)}.$$

7. To explain the physical meaning of f_V, we must have recourse to what the Germans term *Totalverschiebung*, a conception which, so far as I know, has not yet received an English name, although it is an important conception in strain.

Let P_0 and Q_0 be any two points of the unstrained medium separated by a small elementary distance δs_0; obviously δs_0, treated as a vector, gives the position of Q_0 *relative* to P_0. It may be termed the relative-position-vector or simply the *position-vector*. Suppose the body strained, and P_0 and Q_0 to to be now in positions P and Q, the position-vector will now be δs, say. Thus, there has been a change in the position-vector which is represented by the vector difference $\delta s - \delta s_0$, this vector difference per unit of original distance between P_0 and Q_0 is termed the *Totalverschiebung*. It must be carefully distinguished from the *stretch* $\{(\delta s - \delta s_0)/\delta s_0$, where δs and δs_0 are treated as mere *lengths*$\}$ which is a *scalar* quantity. The *Totalverschiebung* is a vector having a definite direction for each direction round a point in the unstrained body. It is obviously a very proper measure of *change of relative position due to strain*. For the purposes of the present paper the *Totalverschiebung* will be spoken of as the *strain-vector*, and represented by f.* Thus the vector f with the tensor P_0Q_0 gives the relative position after strain of the points P and Q. This relative position depends partly on the stretch and partly on the slide and rotational components as giving change of angle, and therefore could be expressed in terms of the strain and twist-components $s_x, s_y, s_z, \sigma_{yz}, \sigma_{zx}, \sigma_{xy}, \tau_{yz}, \tau_{zx}, \tau_{xy}$, but to so express it is clumsy and possesses no advantage for merely physical description.

Let us now find the value of the strain-vector in any direction at any point of an elastic medium.

Let P_0Q_0 be the vector δr with the direction cosines l_1, m_1, n_1, then the coordinates of P (if originally x, y, z) will become

$$x+u, \quad y+v, \quad z+w,$$

and of Q

$$x+l_1\delta r+u+\frac{du}{dr}\delta r,$$

$$y+m_1\delta r+v+\frac{dv}{dr}\delta r,$$

$$z+n_1\delta r+w+\frac{dw}{dr}\delta r,$$

---

* The *strain-vector* includes the effect of a pure rotation, such pure rotation although not effecting the *stresses* ought to be considered as a part of the strain. See Thomson and Tait's *Nat. Phil.*, Part I., §§ 182–3.

where of course

$$\frac{d}{dr}=l_1\frac{d}{dx}+m_1\frac{d}{dy}+n_1\frac{d}{dz}.$$

Hence PQ has for projections on the axes

$$l_1\delta r+\frac{du}{dr}\delta r,\ m_1\delta r+\frac{dv}{dr}\delta r,\ n_1\delta r+\frac{dw}{dr}\delta r,$$

or, if P were moved back to P_0 without any rotation of the medium, the vector Q_0Q would have for components

$$\frac{du}{dr}\delta r,\ \frac{dv}{dr}\delta r,\ \frac{dw}{dr}\delta r.$$

Thus we obtain the components of the strain-vector by dividing these by P_0Q_0 or δr, or the strain-vector in direction δr is given by

$$\frac{du}{dr},\ \frac{dv}{dr},\ \frac{dw}{dr}.$$

8. Returning to F_3', we note that

$$X=l'\frac{du}{dx}+m'\frac{du}{dy}+n'\frac{du}{dz}$$
$$=\frac{du}{dr},$$

if r be the direction of the shift-velocity V.

Thus X, Y, Z are the components of the strain-vector in the direction of the shift-velocity V. If we represent the strain-vector by f_r in direction r, and that in direction V by f_V, it is obvious that we have found for F_3' the value given in equation (xi), with a perfectly clear physical meaning for f_V.

Thus F_3' corresponds to a flow of energy in the direction of the strain-vector for the direction of the shift-velocity and of magnitude equal to 2μ (shift-velocity) × (magnitude of strain-vector for direction of shift-velocity) per unit area of the surface.

9. We are now in possession of a complete analysis of the flow of energy in any vibrating elastic medium. It flows in three directions: (*a*) in the direction of the shift-velocity, (*b*) in the direction of the strain-vector (*i.e.* the absolute displacement of one point relative to a second) for the direction of the shift-velocity, (*c*) perpendicular to the direction of the shift-velocity and to the resultant twist.

The first flux (*a*) is that which we have termed 'pressural energy' flux and is represented per unit area per unit of shift-velocity by the product of the dilatation coefficient and the dilatation. The second flux (*b*) has been termed that of 'strain-vector energy' and is represented by twice the product of the slide-modulus and the strain-vector for the direction of the shift-velocity per unit area per unit of shift-velocity. The third flux (*c*) is represented by twice the product of the slide-modulus into the twist component perpendicular to the shift-velocity per unit area per unit of shift-velocity. If a spectator stand with his back to the twist axis and face the shift-velocity so that the twist gives a right-handed screw-point towards his head, then this flow of energy is from *his left- to his right-hand.*

The "flow of twist-energy" vanishes for a purely pressural wave. For a system of purely tranverse waves the dilatation will be zero, further if the wave-front be taken as the surface over which we are considering the flux of energy, the strain-vector for the direction of the shift-velocity will lie in the wave-face, and therefore there will be no flow F_3' as well as no flow F_1 across an element of that face. F_2 will be the only flow of energy across the wave-face, and this, as we have seen, obeys the law indicated in Poynting's Theorem.

10. A remark may be made in conclusion on the nature of the surface S. The expression for the energy in equation (i) is really a measure of the energy in that portion of the elastic medium which was in the unstrained condition within the surface S. This portion of the medium now occupies the surface S', which is the form taken by the original S considered as a surface fixed in the medium. Properly then to find the flow of energy across S, we ought not only to take the change in the energy in S', but also in the amount of energy in the material of the medium which has flowed across S and now lies between S and S'. But this amount of energy between S and S', since S and S' are only separated by distances of the order of the shifts, is of the third order in small quantities, and its time variation is thus negligible if we retain only quantities of the second order. It was thus legitimate to treat the flow as if it were across the surface S fixed in space and not the flow across S', a surface moving with the material and coinciding with S in the unstrained condition of the medium.

University College, London,
March, 1889.

ON THE REDUCTION OF A BILINEAR QUANTIC OF THE n^{TH} ORDER TO THE FORM OF A SUM OF n PRODUCTS BY A DOUBLE ORTHOGONAL SUBSTITUTION.

By Professor *Sylvester*.

A HOMOGENEOUS lineo-linear function in two sets of variables

$$x, y, \dots z;\ u, v, \dots w$$

will contain n^2 terms: two independent orthogonal substitutions performed on the two sets will introduce twice $\frac{1}{2}n(n-1)$ disposable constants, and by a suitable choice of these, n^2-n terms of the transformed function may be made to vanish so as to leave a sum of products of the new $x, y, \dots z$ paired with the new $u, v, \dots w$: it will of course be found in general impossible to obliterate *any* arbitrarily chosen (n^2-n) terms in the transformed function; since if in the n remaining products one letter of one set were combined with more than one of the other set, this would (by means of a further superimposed orthogonal substitution) be equivalent to taking away more than (n^2-n) terms by means of only (n^2-n) disposable constants. It is very easy to effect the transformation indicated by a method very analogous to that of reducing a quadric in n variables by an orthogonal substitution to its canonical form, and to show *à posteriori* that the substitutions are always, real in this case as in the other, when the original coefficients are real; but it will, I think, (although not necessary) be found interesting and instructive to prove *à priori* the latter assertion by a similar method to that applied to Quadrics in the last number of the *Messenger*. I will begin then with this proof, reserving the complete solution of the problem to the end of the article. The leading idea in this as in the preceding article is to regard a finite orthogonal substitution as the product of an infinite number of infinitesimal ones.

1°. For $\qquad axu + \alpha xv + \beta yu + byv.$

Let $x, y;\ u, v$ become $x+\varepsilon y,\ -\varepsilon x + y;\ u+\lambda v,\ -\lambda u + v$ respectively, then

$$\delta\alpha = a\lambda - b\varepsilon, \quad \delta\beta = a\varepsilon - b\lambda,$$

$$\alpha\delta\alpha + \beta\delta\beta = (a\alpha - b\beta)\lambda + (a\beta - b\alpha)\varepsilon.$$

Hence $\alpha^2+\beta^2$ may be made to decrease unless $a=0$, $b=0$, or $\alpha=0$, $\beta=0$, or $\frac{a}{b}=\frac{\alpha}{\beta}=\pm 1$, in which case since

$$(a+\alpha)(x+y)(u+v)+(a-\alpha)(x-y)(u-v)$$
$$=2a(xu+yv)+2\alpha(xv+yu),$$
$$(a-\alpha)(x+y)(u-v)+(a+\alpha)(x-y)(u+v)$$
$$=2a(xu-yv)+2\alpha(xv-yu),$$

the form is immediately canonizable.

Hence in the infinite succession of infinitesimal orthogonal substitutions (equivalent to a single one) either a and b or α and β must vanish simultaneously, on which supposition the form is canonical or else it is reducible to the canonical form by a second finite orthogonal substitution.

Let us now proceed to the case of a *ternary* bilinear form in x, y, z; u, v, w.

I suppose by the previous case the form to be deprived of two terms, and that we have to deal with the form

$$axu+byv+fxw+guz+hyw+kvz+czw.$$

Lemma. If $f=0$, $g=0$, or $h=0$, $k=0$ the above form is reducible by the previous case. Also if $a^2=b^2$ and $f=0$, $h=0$, or $g=0$, $k=0$, or $a^2=b^2$ and $\left(\frac{f}{h}\right)^2=\left(\frac{g}{k}\right)^2$ the form is reducible to the previous case by a single additional finite orthogonal transformation.

For the sake of brevity I leave the proof to my readers.

Introducing now two infinitesimal orthogonal substitutions with parameters $\varepsilon, \eta, \theta$; λ, μ, ν,* we obtain the variations

$$\delta f=a\mu-h\varepsilon-c\eta,\quad \delta h=b\nu+f\varepsilon-c\theta,$$
$$\delta g=a\eta-k\lambda-c\mu,\quad \delta k=b\theta+g\lambda-c\nu,$$

also in order to keep the coefficients of xv, yu at *null*, we must have

$$a\lambda-b\varepsilon-f\nu-k\eta=0,$$
$$-b\lambda+a\varepsilon-g\theta-h\mu=0.$$

---

* The positive values of the parameters in each system are supposed to belong to the upper, and the negative values to the lower half of each orthogonal matrix.

From the previous equations we obtain

$$f\delta f + g\delta g + h\delta h + k\delta k$$
$$= (af - cg)\,\mu + (bh - ck)\,\nu + (ag - cf)\,\eta + (bk - ch)\,\theta.$$

1°. Suppose $a^2 - b^2$ *not* zero; then μ, ν, η, θ will be independent and their coefficients cannot all become zero unless $f^2 = g^2$ and $h^2 = k^2$, or else $f = 0$ and $g = 0$, or $h = 0$ and $k = 0$, on either of which suppositions the form becomes canonizable by virtue of the Lemma.

2°. Let $a^2 = b^2$. Then we must have

$$f\nu + k\eta \pm (g\theta + h\mu) = 0,$$

which I shall satisfy by making $f\nu \pm g\theta = 0$, $k\eta \pm h\mu = 0$. Hence

$$\Sigma f\delta f = \{(af - cg)\,k \mp (ag - cf)\,h\}\,\rho + \{(ah - ck)\,g \mp (ak - ch)f\}\,\tau,$$

ρ, τ being two arbitrary infinitesimals.

Therefore $\Sigma f\delta f$ may be made negative unless the multipliers of ρ and τ are both zero, in which case by addition or subtraction we obtain $fk = gh$; consequently two out of the four variables f, g, h, k are zero, or else $\dfrac{f}{h} = \dfrac{g}{k}$, and on either of these suppositions the transformed function may be canonized by virtue of what has been proved in the case of two biliteral sets, or may by a finite orthogonal substitution be brought to a form so canonizable.

Hence it is clear that either f, g, h, k may all be made to vanish, or else we must pass through a form known to be canonizable. This is the proof for a bilinear function of triliteral sets, which may be easily extended to a bilinear function of n-literal sets.

I will now give the method for effecting the reduction which is thus proved to be always capable of being effected by real substitutions.

Let $\Sigma a_{r,s} x_r y_s$ be the given bilinear function B.

Then $\Sigma \left(\dfrac{dB}{dy_s}\right)^2$ which is an orthogonal invariant of B quâ the y's, is a Quadratic function of the x's, which will have an orthogonal substitute *quâ* the x's of the form $\Sigma\,[\lambda_r x_r^2]$.

If then B is reducible by a double orthogonal substitution to the form $\Sigma\,[\theta_r x_r y_s]$, we must have $\Sigma\,[\theta_r x_r]^2$ orthogonally equivalent to $\Sigma\,[\lambda_r x_r^2]$, and this can only be the case when the θ's are respectively (in any order) the squares of the λ's.

The θ's I call the Canonical Multipliers to B.

This gives rise to the following rule:

Form the Matrix $[m]$.

$$\begin{array}{l}
a_{1,1}, \quad a_{2,1}, \ldots a_{n,1}, \\
a_{1,2}, \quad a_{2,2}, \ldots a_{n,2}, \\
\ldots\ldots\ldots\ldots\ldots\ldots, \\
a_{1,n}, \quad a_{2,n} \ldots a_{n,n}.
\end{array}$$

From this derive a Matrix $[M]$, a *false* square of $[m]$, obtained by multiplying each line in it by *all* the lines (according to Cauchy's rule, in fact, for the multiplication of *Determinants*). Then the latent roots of $[M]$ are the squares of the Canonical Multipliers to B.

But if instead of $\Sigma\left(\frac{dB}{dy_s}\right)^2$ we take $\Sigma\left(\frac{dB}{dx_r}\right)^2$ and deal with it in like manner, we shall obtain a matrix $[n]$, such that $[m]$ and $[n]$ are transverse to each other, the lines and columns of the one being the columns and lines of the other: the Cauchian Square of $[n]$ will give rise to a matrix $[N]$ different from $[M]$ but having the same latent roots: in fact the coefficients of the equation to the latent roots alike of m and of $[n]$ with the signs in the alternate places changed will be unity, the sum of the squares of all the terms in $[m]$ or $[n]$, the sum of the squares of the minors of the 2nd, 3rd, ... orders in $[m]$ or $[n]$; and finally the last coefficient will be the square of the determinant to $[m]$ or $[n]$: so that we shall obtain as we ought the same set of canonical multipliers whichever matrix $[M]$ or $[N]$ we employ; but in order to obtain the substitutions which must be impressed on the x set and the y set to arrive at the Canonical form in which only n products appear we shall want both $[M]$ and $[N]$. Let me, however, pause for a moment to call attention to the interesting fact that the sum of the squares of the coefficients in B by virtue of being a coefficient of the latent function to $[M]$ or $[N]$ is necessarily a bi-orthogonal invariant to B; so, too, all the other coefficients in this function are such invariants: and among them the last, which is the square of the determinant to $[m]$ or $[n]$. Thus then this determinant (which may be termed the discriminant) is an invariant alike for the two theories; viz. the better known one in which the x set and the y set are subjected to the same general substitution, and the one here considered where these sets are subjected to two independent orthogonal substitutions.

In either theory the vanishing of the discriminant is the signal of the Canonical form becoming short of one term.

It is also proper to notice that the latent roots of $[M]$ or $[N]$, which by virtue of $[M]$ and $[N]$ being symmetrical matrices are necessarily real, are for these particular forms of $[M]$ and $[N]$ *positive* as well as real since the coefficients with the alternate signs changed are all positive, being the sums of squares of real numbers.

To complete the solution it remains to find the two canonizing orthogonal matrices, but these are known by the ordinary theory for quadrics: thus the x substitution will be that which canonizes $[M]$ and the y substitution that which canonizes $[N]$.

Conversely, if $[M]$ and $[N]$ are supposed given, we shall know the linear functions of the x's which substituted for $x_1, x_2, \ldots x_n$ and the linear function of the y's which substituted for $y_1, y_2, \ldots y_n$, such that $\Sigma\lambda_1^{\frac{1}{2}}x_1y_1$ shall be identical with B, the λ's being the latent roots common to $[M]$ and $[N]$. There will be 2^n systems of values represented by $\lambda_1^{\frac{1}{2}}, \lambda_2^{\frac{1}{2}}, \ldots \lambda_n^{\frac{1}{2}}$: thus then 2^n matrices transverse to one another can be found such that their *false* squares shall be respectively identical with any two given symmetrical matrices having the same latent roots, and we are thus enabled indirectly, through the theory of bi-orthogonal canonization, to obtain the solution of a problem which intrinsically has or seems to have nothing to do with orthogonal or other transformation.

It is worthy of observation that this problem of finding the so-to-say *false* square root common to two given symmetrical matrices having the same latent equation, admits of precisely the same number (2^n) solutions as the problem of finding the true square root of one general matrix. For if $[M]$ be any given matrix of order n and $[1]$ represents the unit matrix of that order, namely the matrix all of whose terms are zeros except those in the principal diagonal which are units, we know by virtue of a general theorem that calling $\lambda_1, \lambda_2, \ldots \lambda_n$ its n latent roots, each true square root of $[M]$ is represented by

$$\Sigma\lambda_1^{\frac{1}{2}}\frac{([M]-\lambda_2[1])([M]-\lambda_3[1])\ldots([M]-\lambda_n[1])}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)\ldots(\lambda_1-\lambda_n)}.$$

New College, Oxford,
5th April, 1889.

THE PARABOLIC TRAJECTORY.

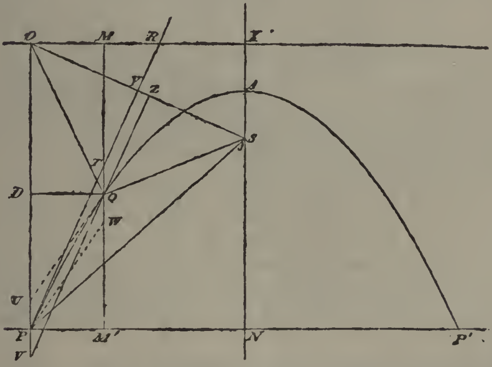
By *A. G. Greenhill.*

The important theorem in Dynamics, that the path of a projectile is a parabola, in the absence of any resisting medium, was first demonstrated by Galileo, and published in his *Dialoghi delle nuove Scienze*, *Giornata quarta*, *De motu projectorum*, Leiden, 1638; although previously various guesses had been made as to the path being some kind of of conic section. (Tartaglia, *Nova Scientia*, 1550; *Quesiti et Inventioni diversi*, 1554; Digges, *New Science of Great Artillerie*, 1591).

Galileo guessed that the trajectory was a parabola from observing the track made by a brass sphere rolling on an inclined plane. His guess, that the catenary curve made by a uniform chain is a parabola, made in a similar experimental manner, was not so successful.

1. Galileo's original demonstration is similar to that which is still found in treatises on Elementary Mechanics; the body is supposed to be projected from a point P with given velocity V in the given direction PT. Then in the time t the body would in the absence of gravity have reached a point T, such that $PT = Vt$ (Fig. A).

Fig. A



But, in the same time t, Galileo asserts that gravity will cause the body to have dropped to a point Q vertically below T, at a depth $TQ = \frac{1}{2}gt^2$, the same depth PV a body would have fallen in the time t if let fall from P.

2. PT was called formerly the *violent motion* of the projectile (*motus violentus*), and TQ the *natural motion* (*motus naturalis*), while the curve PQ, the combination of the violent motion PT and the natural motion TQ, was called the *motus mixtus.*

According to ancient theories of Gunnery the trajectory was composed of (i) the *motus violentus*, where the velocity was so great that the body flew in a straight line, the extent of the *motus violentus* being the *point blank range*, (ii) the *motus mixtus*, the curvilinear portion of the trajectory, (iii) the *motus naturalis*, in which the body fell vertically downwards, as is approximately realized in the neighbourhood of the vertical asymptote of the trajectory, when the resistance of the air is taken into account.

3. Having now

$$PT = Vt, \quad TQ = \tfrac{1}{2}gt^2,$$

the elimination of t leads to the invariable relation for all points on the trajectory PQ,

$$\frac{PT^2}{TQ} = \frac{V^2t^2}{\frac{1}{2}gt^2} = \frac{2V^2}{g} = 4\,OP,$$

if PO is measured vertically upwards and equal to $\frac{1}{2}V^2/g$, the vertical distance the body would have to fall to acquire the velocity V at P.

4. The length OP is called by Galileo the *impetus* of the body at P. The word *impetus* might with great advantage be revived in modern dynamical teaching: thus the product of the weight of the projectile W in pounds and of the impetus in feet is the energy in foot-pounds of the projectile; and the greatest range on a horizontal plane (called the *potential random* by Robins) is twice the impetus.

Taking Hutton's Law, that

$$V = 1600\sqrt{\frac{2P}{W}},$$

giving V the velocity in ft. per sec. imparted to a shot weighing W lb. by a charge of P lb. of powder, so that the velocity is 1600 when the charge is half the weight of the shot, then the impetus

$$h = \tfrac{1}{2}V^2/g = 80000P/W,$$

and the available energy in foot-tons per lb. of powder is

$$\frac{WV^2}{2g \times 2240P} = \frac{Wh}{2240P} = \tfrac{80000}{2240} = 35\tfrac{5}{7}.$$

The strength of modern powder is, however, more than double this, so a better law is

$$V = 1600\sqrt{\frac{5P}{W}} \text{ or } 1600\sqrt{\frac{6P}{W}},$$

giving a velocity of 1600 feet per second when the charge of powder is from one-fifth to one-sixth the weight of the shot.

5. We have now

$$\left.\begin{aligned} PT^2 &= 4\,OP.\,TQ \\ \text{or}\quad QV^2 &= 4\,OP.\,PV \end{aligned}\right\} \text{..................}(1);$$

and *therefore the curve PQ is a parabola*, to use the argument of the ordinary elementary text-books.

But this argument refers the student to a remote and difficult property of the parabola, usually given as a proposition at the end of the chapter on the parabola in a treatise on Geometrical Conics; and thus requires a definite amount of the study of Conic Sections treated geometrically on the part of the student in order to be properly understood and appreciated.

6. Galileo avoids this difficulty to a certain extent by supposing the body to be projected from a given point A in a horizontal direction, with given velocity U suppose; and thus in a similar manner as before shows that

$$NP' = PN = Ut, \quad AN = \tfrac{1}{2}gt^2;$$

and therefore

$$PN^2 = \frac{2U^2}{g}AN = 4AS.AN,$$

for any point P on the trajectory, reached in the time t from the vertex A; and thus obtains the fundamental property of the parabola as defined by the Greek Geometers, from which the curve receive the name παραβολή; the ellipse (ἔλλειψις), and hyperbola (ὑπερβολή) being so called because PN^2 is less or greater than some constant multiple of AN on these curves, and not originally from any reference to the cone of which these curves are sections (Taylor, *Ancient and Modern Geometry of Conics*, XXV.).

7. Galileo, however, shows that the relation that PN^2 bears to AN a constant ratio is a property of the section of a right circular cone made by a plane parallel to a generating line, by taking a figure similar to figure (B); then since

$$PN^2 = L'N.NL,$$

and NL is constant, while $L'N$ varies as AN; therefore PN^2 varies as AN.

8. The most usual definition of a parabola now given in an elementary treatise on Geometrical Conic Sections is "the locus of a point which moves so that its distance from a given fixed point, called the *focus*, is always equal to its distance from a given straight line, called the *directrix*"; and it will render unnecessary a separate study of Geometrical Conics on the part of the student if we can translate the equation (i) of Galileo into the equation expressing the property of the curve according to this definition.

This can easily be effected by converting the proofs of Geometrical Conics, say Prop. IV., Chap. III., of Taylor's *Ancient and Modern Geometry of Conics*; Prop. IX. of Taylor's *Elementary Geometry of Conics*; or of Besant's or Richardson's *Geometrical Conic Sections*.

9. Draw the horizontal straight line OX through O; then, as is well known, this will be the directrix of the parabola (Fig. A).

Now make the angle TPS equal to the angle TPO, and PS equal to PO; then S will be the focus of the parabola; so that if we can prove $SQ = QM$, where M is the point in which QT meets the directrix OX, we have proved that the trajectory is a parabola, according to the usual definition just given.

This can easily be done; for if PT and the parallel VQ are produced to meet SO in Y and Z, and if QD is drawn perpendicular to OP, then since from Galileo's equation (1)

$$QV^2 = 4\,OP.PV,$$

it follows from the property of the similar triangles QDV, OPY, OVZ, that

$$QD^2 = 4\,OY.YZ$$

$$= (OY + YZ)^2 - (OY - YZ)^2$$

(since Y is the middle point of SO)

$$= OZ^2 - SZ^2$$

$$= OQ^2 - SQ^2.$$

Therefore

$$SQ^2 = OQ^2 - QD^2 = OQ^2 - OM^2 = QM^2,$$

or

$$SQ = QM,$$

so that PQ is a parabola, with focus S and directrix OM, according to the definition that the locus of a parabola PQ is such that the focal distance SQ of any point Q is equal to the distance QM of Q from the directrix.

10. The tangent PT at the point of projection P bisects the angle SPO, and is perpendicular to SO; and similarly the line UQ at any other point Q bisecting the angle SQM, and therefore perpendicular to SM, is a tangent to the curve at Q.

For $SU = UM$, and SU is therefore greater than UO, and U, any point on UQ, is therefore outside the parabola, except when U coincides with Q (Taylor, *Ancient and Modern Geometry of Conics*, Prop. VI., p. 51. *The Doctrine of Projectiles*, by William Starrat, Dublin, 1746; Theorem IX., p. 48).

11. But suppose we wish to go a step further back and prove that the curve PQ is the plane section of a right circular cone, having a vertical generating line, made by a vertical plane.

Describe any sphere on a diameter SK, touching this vertical plane at S (Fig. B); then it is well known that a right circular cone can be drawn, touching this sphere along its section made by the plane XOK through the directrix XO and the point K, and that PQ will be the curve of section of this cone made by the vertical plane through XO.

Fig. B has been drawn, in true perspective, by Mr. A. G. Hadcock, R.A.

The distance of the vertex of the cone from the plane must be equal to the diameter of the sphere for the section of the cone made by the plane to be a parabola.

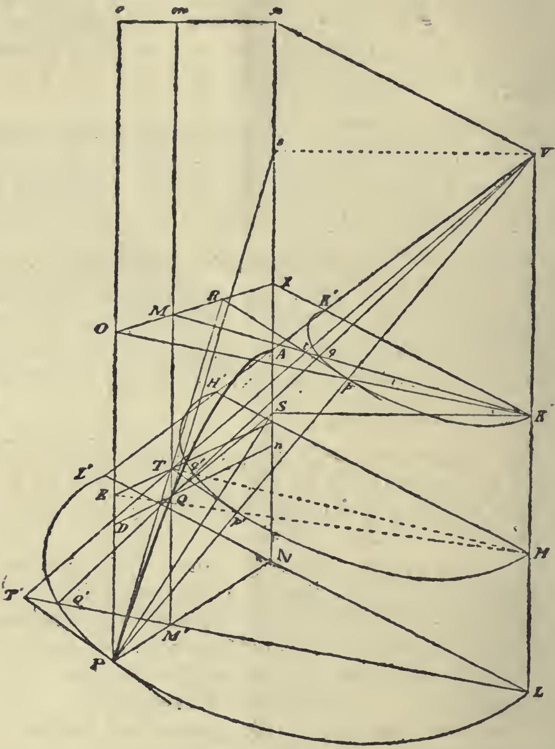
If the distance is greater than the diameter, the section will be an ellipse; if less, a hyperbola; and the point of contact of the sphere with the plane will always be a focus of the conic section.

This is well seen by observing the shadow thrown by a billiard ball on a table, when a light is placed in different positions at different distances from the table.

12. The first thing to notice is that the locus of V, the vertex of this cone, for a given parabola PQ is the *focal parabola*, having its vertex at S, and focus at A, the vertex of the parabola PQ. For

$$AV = Ax = AX + Xx = AS + Ss,$$

Fig. *B*



where Vs is perpendicular to SX, and the plane Vxo has been drawn through V parallel to the plane KXO, and therefore by the property of the parabola SV,

$$Vs^2 = 4AS . Ss, \text{ or } SK^2 = 4AS . VK;$$

as is otherwise evident from the fact that

$$AS = AK' \text{ and } VK = VK',$$

so that

$$AS . Vk = AK' . K'V$$

$$= \text{square on the radius of the sphere} = \tfrac{1}{4} SK^2.$$

Now if VP meets the plane XOK in p, then

$$Pp = PS = PO,$$

Pp and PS being tangents to the sphere; also

$$pV = VK = Oo;$$

therefore

$$PV = Po.$$

13. To prove that Q also lies on the cone, we must prove that $QV = Qm$, where m is the point in which QM produced meets ox; whence it follows that QV is a generating line of the cone.

We have just proved that $QS = QM$; and therefore employing this property,

$$\begin{aligned} QV^2 &= Qs^2 + sV^2 \\ &= QS^2 + Ss^2 + 2Ss\,.\,Sn + 2XS\,.\,Ss \\ &= QS^2 + Ss^2 + 2Ss\,.\,Xn \\ &= (QM + Mm)^2 = Qm^2, \end{aligned}$$

or $$QV = Qm;$$

thus proving that QV is a generating line of the cone; and therefore Q is a point on the section of the cone made by the vertical plane through OX.

14. Suppose, however, we wish to translate immediately Galileo's equation (1),

$$PT^2 = 4OP\,.\,TQ$$

into a theorem showing that Q lies on the cone; to do this we draw the planes through T and P parallel to the plane XOK, thus cutting the cone in circles.

Then if these planes are cut by VK in H and L, by VP in p' and P, by VT in T and T', and by VQ in q' and Q' respectively, we have in the first place $Tp' = TE$, the perpendicular drawn from T upon PO, in consequence of the equality of the triangles TPO, TPS, TPp.

By the property of similar triangles,

$$\frac{PT^2}{TE^2} = \frac{OP^2}{OY^2} = \frac{PS^2}{SY^2} = \frac{SY^2}{SA^2} = \frac{OP}{SA};$$

and therefore, by equation (1),

$$\begin{aligned} TE^2 = Tp'^2 &= 4AS\,.\,TQ \\ &= 4AS\,.\,\frac{VH}{Hq'}\,.\,Tq', \end{aligned}$$

from the similar triangles TQq', VHq'; and therefore

$$\begin{aligned}\frac{Tp'^2}{Tq'} &= \frac{4AS.VH}{Hq'}\\ &= \frac{4AS.VK.VH}{Hq'.VK}\\ &= \frac{KS^2.VH}{Hq'.VK}\\ &= \frac{KK'.KX.VH}{Hq'.VK}\\ &= \frac{Kp.KO.VH}{Hq'.VK}\\ &= \frac{Kp.KO}{Hq'}\cdot\frac{Hp'}{Kp}\\ &= \frac{Hp'.HE}{Hq'}\\ &= \frac{HT^2-Tp'^2}{Hq'},\end{aligned}$$

since $$TE=Tp'.$$

Therefore $$\frac{Tp'^2}{HT^2-Tp'^2}=\frac{Tq'}{Hq'},$$

or, *componendo*,

$$\frac{Tp'^2}{HT^2}=\frac{Tq'}{HT},$$

or $$Tp'^2=Tq'.TH,$$

which proves that q' lies on the circle $Hp'H'$; and therefore proves that VQ is a generating line of the cone.

We might also have translated equation (1) by the property of similar triangles into the relation

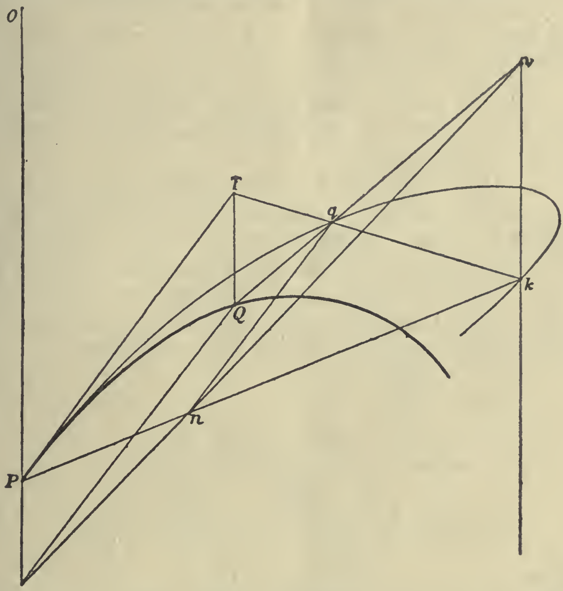
$$tp^2=tq.tK,$$

or $$T'P^2=T'Q'.T'L,$$

thus proving that q and Q' lie on the cone.

15. We can prove that the curve PQ defined by equation (1) is the section made by a vertical plane of a cone of the second degree, not necessarily a right circular cone, but having oblique circular sections, in Clifford's manner (*Elements of Dynamic*, p. 39), by taking any vertex v, and drawing

through it the vertical straight line vk, and then drawing any fixed plane through PT meeting vk in k. (Fig. C).



Then if the plane vQn is drawn through vQ parallel to PT, meeting Tk and Pk in q and n, qn will be parallel to PT; and by similar triangles

$$\frac{qn^2}{nk^2} = \frac{PT^2}{Pk^2} = \frac{4OP.TQ}{Pk^2}$$

$$= \frac{4OP.vk}{Pk^2}\,\frac{Tq}{qk}$$

$$= \frac{4OP.vk}{Pk^2}\,\frac{Pn}{nk},$$

or
$$\frac{qn^2}{Pn.nk} = \frac{4OP.vk}{Pk^2},$$

a constant ratio for different positions of Q and q, so that the curve Pq in the plane Pqk is an ellipse, and therefore the cone with vertex v and plane section PQ stands on the elliptic base Pqk, and is therefore a cone of the second degree.

16. If the plane PTk is drawn through PT perpendicular to the vertical plane through PQ, then Pk is a principal axis of the ellipse Pqk; and if in addition we make

$$Pk^2 = 4OP.vk,$$

by taking v on the parabola Pv having its focus at O, vertex at P, and lying in the vertical plane OPv perpendicular to the vertical plane of PQ, then the ellipse Pqk will become a circle, and sections parallel to it will be circular sections of the cone vPk.

17. The impetus at the point of projection P being OP, suppose it is required to prove that the impetus at any other point Q is MQ (Fig. A).

To do this, produce TQ vertically downwards to W, making $QW = TQ$; then since the velocity v at Q is compounded of the original velocity of projection V in TQ, and of a vertical velocity gt, it follows from the triangle of velocities that the direction of motion, and therefore the tangent at Q, is parallel to PW; or the tangent at Q meets OP in a point U, such that $UP = PV$; also that

$$\begin{aligned} \frac{v^2}{V^2} &= \frac{PW^2}{PT^2} \\ &= \frac{PT^2 + TW^2 - 2TW.TM'}{PT^2} \\ &= 1 - \frac{4TQ.QM'}{PT^2} = 1 - \frac{QM'}{OP} = \frac{QM}{OP}; \end{aligned}$$

and since $$\tfrac{1}{2}V^2/g = OP,$$

therefore $$\tfrac{1}{2}v^2/g = QM,$$

the impetus at Q.

18. The directrix OM may be taken to be the *hodograph* of the trajectory, the velocity at any point P or Q being perpendicular to and proportional to SO or SM respectively.

The horizontal component of the velocity in the trajectory being constant, it follows that the velocity of M in the hodograph is constant; and therefore the acceleration in the trajectory PQ is constant and perpendicular to OM, and therefore vertical.

Drawing the normal PG at P, the horizontal and vertical components of the velocity at P, viz. U and gt are by the triangles of velocities in the ratio of NG to PN; and since $PN = Ut$, therefore

$$NG = PN.U/gt = U^2/g,$$

a constant; this is the proof given by Prof. Hart in the *Messenger of Mathematics*, vol. x., p. 64, that the subnormal in the trajectory is constant, a fundamental property of the parabola.

THE SOLUTION OF A SPECIAL CASE OF THE PROBLEM OF THE ESTABLISHMENT OF A CORRELATION BETWEEN TWO PLANE FIGURES.

By *J. Brill, M.A.*, St. John's College.

1. Schwarz and Christoffel have solved the problem of the establishment of a correlation between the region inside a closed convex rectilinear polygon and a half-plane; and Harnack* has adapted the method of Schwarz to the establishment of a correlation between the region outside such a polygon and a half-plane, the correlative of infinity in the diagram containing the polygon being a point in the half-plane at a finite distance from the origin. The problem that I propose to solve in this paper is that of the establishment of a correlation between the region outside the polygon and a strip of the half-plane, bounded by a finite portion of the axis of x and two straight lines parallel to the axis of y, the infinity of the one diagram being the correlative of the infinity of the other. This is equivalent to the problem of finding the irrotational cyclic motion of a perfect incompressible fluid about a prism having the polygon for its principal section, on the supposition that the motion of the fluid is everywhere continuous.† For the stream function will have the value zero along the perimeter of the polygon, and will increase indefinitely as we pass away from the polygon and go off to infinity; also the velocity potential will be a multiple valued function, being the inverse of some periodic function. Thus the function of a complex variable that establishes the required correlation will divide the plane into a series of strips similar to that described above; and in these strips there will be a set of exactly similar diagrams, each corresponding to the diagram in the other plane, which consists of the stream lines in the given case of fluid motion and their orthogonal trajectories.

We shall effect the solution of our problem with the aid of a method* slightly different from that adopted by Schwarz and Christoffel.

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* *Die Grundlagen der Theorie des logarithmischen Potentiales und der eindeutigen Potentialfunktion in der Ebene*, § 49, Leipzig, 1887.

† This of course will give rise to a solution which, though kinematically possible, is however physically impossible, as it will involve infinite velocities at the corners of the polygon.

2. Suppose that we have a certain function of a complex variable $\zeta = f(z) = \xi + i\eta$, then it follows that

$$\frac{d\zeta}{dz} = f'(z) = \frac{\partial\xi}{\partial x} - i\frac{\partial\xi}{\partial y},$$

and therefore

$$\begin{aligned}\log\frac{d\zeta}{dz} &= \log\left(\frac{\partial\xi}{\partial x} - i\frac{\partial\xi}{\partial y}\right)\\ &= \tfrac{1}{2}\log\left\{\left(\frac{\partial\xi}{\partial x}\right)^2 + \left(\frac{\partial\xi}{\partial y}\right)^2\right\} - i\tan^{-1}\left\{\frac{\partial\xi}{\partial y}\Big/\frac{\partial\xi}{\partial x}\right\}\\ &= \log h - i\vartheta.\end{aligned}$$

If we interpret ξ as the velocity potential and η as the stream function of a case of irrotational fluid motion, then h is the velocity of the fluid at any point, and ϑ is the angle which the direction of motion at that point makes with the positive direction of the axis of x. If, therefore, we express the angle which the tangent at any point to the stream line through it makes with the positive direction of the axis of x, in terms of the coordinates of its point of contact, then the resulting expression will satisfy Laplace's equation.

It is also to be remarked that if $\zeta_1, \zeta_2, \ldots, \zeta_n$ be n functions of a complex variable, and if we write

$$m\log\frac{d\zeta}{dz} = m_1\log\frac{d\zeta_1}{dz} + m_2\log\frac{d\zeta_2}{dz} + \ldots + m_n\log\frac{d\zeta_n}{dz},$$

then ζ will also be a function of a complex variable. And if we interpret $\zeta, \zeta_1, \zeta_2, \ldots, \zeta_n$ as giving rise to cases of fluid motion, using $\vartheta, \vartheta_1, \vartheta_2, \ldots, \vartheta_n$ to denote the angles which the tangents at any the same point to the respective stream lines through it make with the positive direction of the axis of x, then we have

$$m\vartheta = m_1\vartheta_1 + m_2\vartheta_2 + \ldots + m_n\vartheta_n.$$

These considerations will furnish us with material for the solution of our problem.

3. Let $A_1A_2\ldots A_n$ be the given convex rectilinear polygon, and let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be the exterior angles between A_nA_1 and A_1A_2, A_1A_2 and A_2A_3, &c., respectively. Take A_1 as origin and A_1A_2 as the direction of the axis of x.

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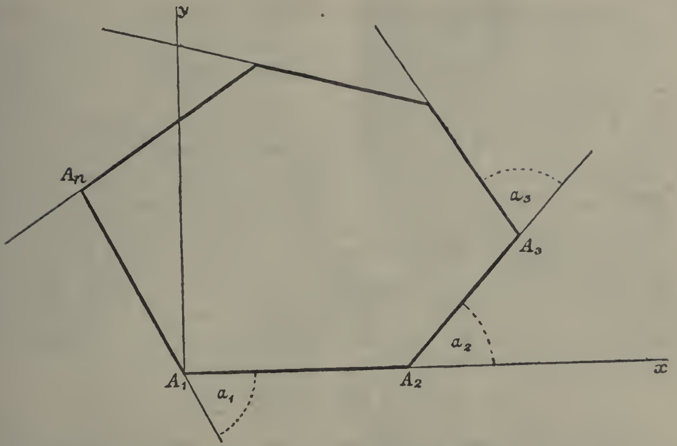
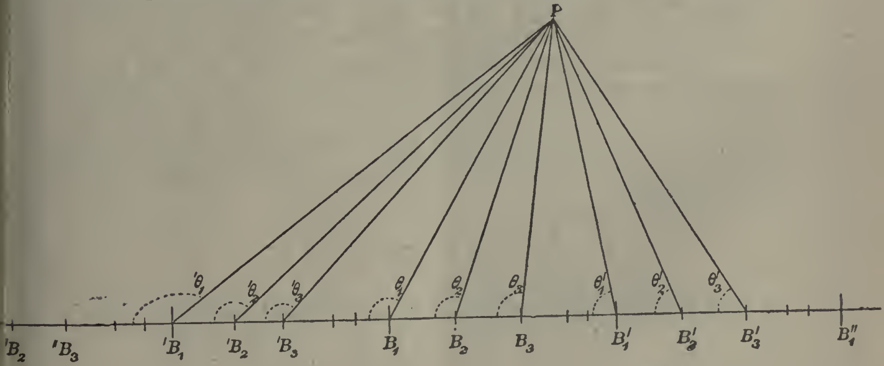
* I made use of this method in a paper entitled "Notes on Conjugate Functions and Equipotential Curves," *Proc. C. P. S.*, VI., pp. 187–199. When I wrote the paper I was not acquainted with the work of Schwarz and Christoffel.



Let the portion B_1B_1' of the axis of ξ be the correlative of the perimeter $A_1A_2\ldots A_nA_1$ of the polygon, the points B_1 and B'_1 corresponding to A_1, and B_2, B_3, &c., corresponding to A_2, A_3, &c. Also take B_1 as the origin of the correlative diagram, and divide the axis of ξ into a series of portions $B_1'B_1''$, $B_1''B_1'''$, &c., $'B_1B_1$, $''B_1{}'B_1$, &c., each equal to B_1B_1'. Divide each of these portions into lengths equal to those into which the portion B_1B_1' is divided by the points B_2, B_3, &c.; and denote the point which corresponds to B_2 in any section by the symbol B_2 affected with the dashes proper to the section to which it belongs, and so on. Take a point P in the portion of the ζ plane above the axis of ξ, and join it with all the points of division of the axis of ξ; and let the angles which these lines make with the negative direction of the axis of ξ be each denoted by the symbol θ affected with the same subscript and dashes as the point at the vertex of the said angle.



Consider the expression

$$\vartheta = \frac{1}{\pi} \Sigma (\alpha_1\theta_1 + \alpha_2\theta_2 + \ldots + \alpha_n\theta_n),$$

the Σ indicating that we are to add all the expressions that can be obtained, of which the one written down is a type. This expression will satisfy Laplace's equation when expressed in terms of ξ and η; and, therefore, also when expressed in terms of x and y. We proceed to discuss the changes in value which it undergoes along the axis of ξ. If we write κ for its value immediately before it reaches B_1, then its value from B_1 to B_2 will be $\kappa + \alpha_1$, that from B_2 to B_3 will be $\kappa + \alpha_1 + \alpha_2$, and so on. Finally, immediately before it reaches B_1', its value will be $\kappa + 2\pi$, since we have

$$\alpha_1 + \alpha_2 + \alpha_3 + \ldots + \alpha_n = 2\pi.$$

In a similar manner we can trace its value all along the axis of ξ, and we shall find that its value at any given point of a specified compartment (*i.e.* the space between two consecutive B_1's) will differ from its value at the corresponding point of the preceding compartment by 2π.

Further, the quantity ϑ will be continuous throughout the portion of the ζ plane above the axis of ξ. However, at places at a finite distance from the origin its expression will contain an infinite constant, but this will be got rid of in the work that follows. Moreover, if we divide the upper portion of the ζ plane into a series of strips by drawing lines through the B_1 points parallel to the axis of η, it is easy to see that the value of ϑ at any point of one of the strips will differ from its value at the corresponding point of the preceding strip by the quantity 2π. For, from symmetry, it is evident that its respective values at the said points will differ from its respective values at the feet of the perpendiculars from those points on the axis of ξ by the same amount; and the values at the feet of the perpendiculars differ by the amount 2π.

Thus we see that ϑ obeys all the conditions that are to be satisfied by the expression for the angle made by the tangent at any point to a stream line in the z diagram with the positive direction of the axis of x. And if we write $\pi - \vartheta_1$ for θ_1, $\pi - \vartheta_2$ for θ_2, and so on, we see that ϑ will differ from the expression

$$-\frac{1}{\pi} \Sigma (\alpha_1\vartheta_1 + \alpha_2\vartheta_2 + \ldots + \alpha_n\vartheta_n)$$

by an infinite constant which is an integral multiple of 2π.

We may suppose this infinite constant to become absorbed in that which ϑ already contains, which is also an integral multiple of 2π, and it will consequently suit our purposes if we write

$$\vartheta = -\frac{1}{\pi}\,\Sigma\,(\alpha_1\vartheta_1 + \alpha_2\vartheta_2 + \ldots + \alpha_n\vartheta_n).$$

Let a_r denote the distance of the point B_r from the origin. Then, if h denote the distance B_1B_1', the distance of $B_r^{(s)}$ from the origin will be $a_r + sh$. Now consider the expression

$$\log\{(\xi - a_r - sh) + i\eta\} = \log\{\zeta - (a_r + sh)\}.$$

The imaginary part of this expression will be

$$i\tan^{-1}\frac{\eta}{\xi - a_r - sh} = i\vartheta_r^{(s)}.$$

Moreover, the imaginary part of the expression will not be altered by the addition or subtraction of any real quantity. We will therefore diminish the value of the expression by the amount $\log sh$, and it will be found that this device will relieve us of all difficulties with regard to the introduction of infinite constants. If therefore we sum all the terms of the type

$$\frac{\alpha_r}{\pi}\log\frac{\zeta - (a_r + sh)}{sh}$$

that can be written down, we shall have a function of a complex variable whose imaginary part is $-i\vartheta$. And if we sum all the expressions of this type that contain a_r, r retaining a particular value, we obtain

$$\frac{\alpha_r}{\pi}\log\left[(\zeta - a_r)\left\{\frac{(\zeta - a_r)^2}{h^2} - 1\right\}\left\{\frac{(\zeta - a_r)^2}{2^2h^2} - 1\right\}\left\{\frac{(\zeta - a_r)^2}{3^2h^2} - 1\right\}\ldots\ ad.\ inf.\right]$$

$$= \frac{\alpha_r}{\pi}\log\left\{\pm\frac{h}{\pi}\sin\frac{\pi(\zeta - a_r)}{h}\right\} = \frac{\alpha_r}{\pi}\log\sin\frac{\pi(\zeta - a_r)}{h} + \text{const.};$$

and it is therefore evident that the solution of our problem will be given by the equation

$$\log\frac{d\zeta}{dz} = \frac{\alpha_1}{\pi}\log\sin\frac{\pi(\zeta - a_1)}{h} + \frac{\alpha_2}{\pi}\log\sin\frac{\pi(\zeta - a_2)}{h} + \ldots + \frac{\alpha_n}{\pi}\log\sin\frac{\pi(\zeta - a_n)}{h} + \log K,$$

where K is some constant that may be complex. And it is to be noted that in this result we have introduced the quantity a_1 for the sake of symmetry, though its value is zero.

Thus we have

$$z = C\int \frac{d\zeta}{\left\{\sin\frac{\pi(\zeta - a_1)}{h}\right\}^{\frac{\alpha_1}{\pi}} \left\{\sin\frac{\pi(\zeta - a_2)}{h}\right\}^{\frac{\alpha_2}{\pi}} \cdots \left\{\sin\frac{\pi(\zeta - a_n)}{h}\right\}^{\frac{\alpha_n}{\pi}}},$$

where $C = 1/K$.

The only remaining point to be noticed is the fact that the infinities of the two diagrams are correlatives. This is evident since the stream function varies from zero to infinity as we pass from a point on the perimeter of the polygon to an infinite distance; also in the correlative diagram we have the axis of $\xi\ (\eta = 0)$ corresponding to the perimeter of the polygon, and at the infinity of the correlative diagram we have η infinite, η corresponding to the stream function of the other diagram.

4. The solution of the problem is not definitely completed until the values of the constants $a_1, a_2, \ldots, a_n$ are determined. In the present state of our knowledge this can only be done in particular cases. If the exterior angles of the polygon be definite fractions of two right angles, then the substitution

$$\lambda = \sin\frac{\pi\zeta}{h}$$

will transform the above integral into an Abelian integral. If the said Abelian integral be of a form whose properties have been studied and are completely known, then the determination of the constants in the given case can be effected.

If we had a method of determining the constants in the general case, then, by making the number of sides of the polygon increase indefinitely and their lengths diminish indefinitely, we could discover the function of a complex variable that establishes a correlation between the region outside a given plane closed curve and a strip such as that discussed above. The given curve, however, would have to be of such a character that its convexity was turned outwards all round its perimeter.

ON AN ARITHMETICAL THEOREM IN PERIODIC CONTINUED FRACTIONS.

By Professor *Sylvester.*

The well-known form of continued fraction for the square root of N, an integer, is

$(a;\ b, c, d, \ldots, d, c, b, 2a;\ b, c, d, \ldots, d, c, b, 2a;$ indefinitely continued)

which, if we denote the type $a, b, c, d, \ldots, d, c, b, a$ by t, may be written under the more convenient form

$$(t, 0, t, 0, t, 0, \ldots\ \textit{ad inf.})$$

If now we use $[t]$ to signify the cumulant of which t is the type, and $[t']$, $[{}^{\backprime}t']$ respectively, the cumulants of the types got by cutting off a from either end and from both ends of t, it is easily shown that *whatever* numbers $a, b, c, \ldots$ represent, the value of the continued fraction $\{(t, 0)^\infty\}$ is $\sqrt{\dfrac{[t]}{[{}^{\backprime}t']}}$, so that if $\{(t, 0)^\infty\}$ represents the square root of an integer, $[t]$ must be divisible by $[{}^{\backprime}t']$.

At first sight one would imagine that it would be a difficult matter to give a rule for determining whether such condition is fulfilled or not by any assigned value of the symmetrical type t, but Mr. C. E. Bickmore, of New College, Oxford, has noticed that the case is quite otherwise, for that if we put t under the form a, τ, a, then, in order that $\{(a, \tau, a, 0)^\infty\}$ may satisfy the requirement of being the square root of an integer, the sufficient and necessary condition is the equivalence

$$2a \equiv (-)^\mu [\tau'] [{}^{\backprime}\tau'] \ (\text{mod.}\ [\tau]),$$

where μ is the number of elements in τ.

Consequently τ may be taken quite arbitrarily, and then an infinite number of values be assigned to a, except in the case where $[\tau]$ is even and at the same time $[\tau']$ and $[{}^{\backprime}\tau']$ are each of them odd.

The proof in my notation is as follows:

Since $t = a, \tau, a$, we have ${}^{\backprime}t' = \tau$, and consequently $\dfrac{[t]}{[{}^{\backprime}t']}$ will be an integer if

$$[a, \tau, a] \equiv 0 \ (\text{mod.}\ [\tau]).$$

Expanding and remembering that $[{}^{\backprime}\tau] = [\tau']$ (the type τ being symmetrical), we obtain

$$a^2[\tau] + 2a[\tau'] + [{}^{\backprime}\tau'] \equiv 0 \text{ (mod. } [\tau]).$$

Hence $$2a[\tau'] + [{}^{\backprime}\tau'] \equiv 0 \text{ (mod. } [\tau]) \ldots\ldots\ldots\ldots\ldots(1),$$

and $$2a[\tau']^2 + [\tau'][{}^{\backprime}\tau'] \equiv 0 \text{ (mod. } [\tau]) \ldots\ldots\ldots(2).$$

But $$[\tau']^2 - [\tau][{}^{\backprime}\tau'] = (-1)^{\mu+1},$$

so that $$[\tau']^2 \equiv (-1)^{\mu+1} \text{ (mod. } [\tau]),$$

and therefore (2) becomes

$$2a \equiv (-)^{\mu}[\tau'][{}^{\backprime}\tau'] \text{ (mod. } [\tau]) \ldots\ldots\ldots\ldots\ldots(3),$$

which is thus shown to be a necessary condition.

It is also a sufficient condition, for multiplying (3) by $[\tau']$ we have

$$2a[\tau'] \equiv (-)^{\mu}[\tau']^2[{}^{\backprime}\tau'] \text{ (mod. } [\tau]),$$

or, since $$[\tau']^2 \equiv (-)^{\mu+1} \text{ (mod. } [\tau]),$$

$$2a[\tau'] \equiv -[{}^{\backprime}\tau'] \text{ (mod. } [\tau]),$$

which is the same as (1).

Suppose now that ${}^{\backprime}\tau'$ is given and that we wish to ascertain if a can be found of such a value that the congruence (3) shall be soluble. This will obviously be the case if $[\tau]$ is odd. It will also be the case if $[\tau]$ is even, provided $[{}^{\backprime}\tau']$ is also even, and only in that case; for, when $[\tau]$ is even, then by virtue of the equation

$$[{}^{\backprime}\tau'][\tau] - [\tau']^2 = \pm 1,$$

$[\tau']$ must be odd.

We have, therefore, to find under what circumstances $[{}^{\backprime}\tau']$ will be odd and $[\tau]$ even; in all other cases but these the congruence (3) will be soluble, and then the most general value of a will be any term in an arithmetical series of which the common difference is $[\tau]$, unless $[\tau]$ and $[{}^{\backprime}\tau']$ are both of them even, in which case the common difference will be $\frac{1}{2}[\tau]$.

I proceed now to give a rule for determining the possible and impossible cases of the solution of (3), to explain the grounds of which the following statement will suffice.

1°. The value of a cumulant is not affected by striking out any even number of consecutive zeros from its type.

2°. The *parity* (*i.e.* the character *quâ* the modulus 2) of any cumulant will not be affected if we strike out 3 consecutive odd terms, whether they occur in the middle or at either extremity. For if t, τ be any two types, the cumulant

$$\begin{aligned} [t, 1, 1, 1, \tau] &= 3[t][\tau] + 2[t'][\tau] + 2[t][{}^{`}\tau] + [t'][{}^{`}\tau] \\ &\equiv [t][\tau] + [t'][{}^{`}\tau] \pmod{2} \\ \text{i.e.} \qquad &\equiv [t, \tau] \pmod{2}, \end{aligned}$$

Also

$$[1, 1, 1, t] = [t, 1, 1, 1] = 3[t] + 2[{}^{`}t] \equiv [t] \pmod{2}.$$

3°. The value of any cumulant in the type of which 1, 0, 1 occurs anywhere is the same as if 2 is substituted for 1, 0, 1; and therefore its parity is not affected if the units on each side of the 0 are omitted.

In what precedes in Nos. (1), (2), (3) the result, to modulus 2, is obviously unaffected if for 0 we write any even and for 1 any odd number.

In order then to determine the parity of $[{}^{`}\tau']$ and of $[\tau]$ we may proceed as follows:

Let τ be any assigned symmetrical type, ${}^{`}\tau'$ will then represent the type divested of its two equal terminals.

Rules—(1) for each even number in ${}^{`}\tau'$ write 0, and for each odd number, 1;

(2) elide any even number of consecutive *zeros*, and any number divisible by 3 of consecutive *units*;

(3) elide any pair of units lying on each side of a zero;

(4) repeat these processes as often as possible;

then, I say, eventually we must arrive at one or other of the six following irreducible types, viz.

( ); 0; 1; 1, 1; 0, 1, 0; 0, 1, 1, 0,*

where ( ) means absolute vacuity; accordingly ${}^{`}\tau'$ may be said to be affected with one or the other of these 6 *characters*.

If now the reduced form of ${}^{`}\tau'$ is 0; 1, 1; 0, 1, 0, $[{}^{`}\tau']$ is even, and the congruence (3) will be soluble. In the other three cases [‘τ’] is odd, but $[\tau]$ will also be odd unless its terminal elements are odd in the case where the reduced form of ‘τ’ is ( ), and even for the reduced forms 1, and 0, 1, 1, 0.

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* Except for the symmetrical form of τ there would be two additional (virtually undistinguishable) reduced forms 0, 1 and 1, 0.

In the following exhaustive table the second column indicates the evenness or oddness of the terminals of τ denoted by e and u respectively.

The third and fourth columns indicate the evenness or oddness (denoted as above) of [‘τ’] and $[\tau]$, along with the character of ‘τ’ in the third column. In the fifth column the answer is given as to the determining congruence being soluble or insoluble, denoted by s and i respectively; and the last column shows whether the common difference of the arithmetical series of the values of either terminal, in the case of solubility, is equal to the modulus $[\tau]$ or its moiety.

Cases.	Terminals.	‘τ’		$[\tau]$	Sol. or Insol.	C.D.
1	e	( )	u	u	s	$[\tau]$
2	u	( )	u	e	i	
3	e	1	u	e	i	
4	u	1	u	u	s	$[\tau]$
5	e	0, 1, 1, 0	u	e	i	
6	u	0, 1, 1, 0	u	u	s	$[\tau]$
7	e	0	e	e	s	$\frac{1}{2}[\tau]$
8	u	0	e	e	s	$\frac{1}{2}[\tau]$
9	e	1, 1	e	u	s	$[\tau]$
10	u	1, 1	e	u	s	$[\tau]$
11	e	0, 1, 0	e	u	s	$[\tau]$
12	u	0, 1, 0	e	u	s	$[\tau]$

The following examples are given to prevent the possibility of mis-apprehension in the application of the Algorithm.

(α) Let

$$\tau = 1, 9, 1, 1, 1, 2, 1, 7, 4, 2, 2, 2, 4, 7, 1, 2, 1, 1, 1, 9, 1,$$

$$\begin{aligned} \text{Then } {}^{\backprime}\tau' &\equiv 1, 1, 1, 1, 0, 1, 1, 0, 0, 0, 0, 0, 1, 1, 0, 1, 1, 1, 1 \\ &\equiv \quad 0, \quad 1, \qquad 0, 1, \quad 0 \\ &\equiv \quad 0, \qquad\quad 0, \quad 0 \\ &\equiv \quad 0 \end{aligned}$$

This corresponds to case (8), which is a soluble one, and accordingly we have from Degen's Table

$$\{(15, \tau, 15, 0)^{\infty}\} = \surd(251),$$

15 being the first term of an arithmetical series whose common difference is $\frac{1}{2}[\tau]$.

(β) Let $\tau = 2, 3, 1, 2, 4, 1, 6, 6, 1, 4, 2, 1, 3, 2$.

$$\begin{aligned} \text{Then} \qquad {}^{\backprime}\tau' &\equiv 1, 1, 0, 0, 1, 0, 0, 1, 0, 0, 1, 1 \\ &\equiv 1, 1, \quad 1, \quad 1, \quad 1, 1 \\ &\equiv \qquad (\ ). \end{aligned}$$

This corresponds to the soluble case (1), and accordingly we find from Degen's Table $\{(10, \tau, 10, 0)^{\infty}\} = \surd(109)$; 10 being the first term of an arithmetical series whose common difference is $[\tau]$.

New College,
June 3, 1889.

ON AN EXTENSION OF A PROBLEM OF PAPPUS'S.

By *Isuruta Kenji*, Tokio, Japan.

LAST summer Prof. Fujisawa, of the Imperial University, Tokio, called my attention to the following geometrical problem, which may be looked upon as an extension of a problem of Pappus's* :—

"Given two straight lines and a point in one plane, to draw a straight line through the given point so that its portion included between the two given straight lines may be equal to a given length."

A solution of this problem may be made to depend upon the following two theorems, which can easily be proved :

(i) The locus of the middle point of a straight line of constant length and included between any two given straight lines is an ellipse.

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* Catalan, *Théorémes et Problémes de Géométrie Élémentaire*, p. 189.

(ii) The locus of the centre of gravity of a triangle formed by a straight line through a fixed point with two given straight lines coplanar with the point is a hyperbola.

The construction is as follows:—Describe an ellipse, (i) taking the constant length equal to two-thirds of the given length; then describe a hyperbola, (ii) the given point being taken as the fixed point. Again, describe another ellipse similar to and concentric with the above ellipse, the ratio of similitude of the former to the latter being 3 : 2. In general the inner ellipse and the hyperbola will intersect in four points. Draw four radii vectors through these points from the intersection of the given straight lines, and let them intersect the outer ellipse in P_1, P_2, P_3, P_4. Join P_1, P_2, P_3, P_4 with the given point O. The straight lines OP_1, OP_2, OP_3 OP_4 have their segments included between the given straight lines equal to the given length.

THE SMALL DEFORMATION OF CURVES AND SURFACES WITH APPLICATION TO THE VIBRATIONS OF A HELIX AND A CIRCULAR RING.

By *J. H. Michell, B.A.*, Scholar of Trinity College, Cambridge.

The small deformation of curves and surfaces.

Suppose that at each point of a curve in space there is drawn a rectangular set of axes according to a definite law.

Let the coordinates of a point P on the curve referred to a fixed set of axes be x, y, z, and let the direction-cosines of the axes at the point P be $(l_1m_1n_1)$, $(l_2m_2n_2)$, $(l_3m_3n_3)$.

Let the rates of twist of these three axes about themselves be θ_1, θ_2, θ_3 respectively.

Let s denote the length of the arc up to $P(x, y, z)$.

Consider a point A on the axis of x at unit distance from the origin O.

The projections of OA on the axes at $P(x, y, z)$ are l_1, l_2, l_3 respectively.

The rate at which A travels along the direction $l_1m_1n_1$ as P moves along the curve is expressed by

$$\frac{dl_1}{ds} - l_2\theta_3 + l_3\theta_1;$$

(Routh, *Rigid Dynamics*, II., p. 3)

but this is zero; and therefore

$$\frac{dl_1}{ds} = l_2\theta_3 - l_3\theta_2.$$

Writing down two similar results in like manner, we have the three equations*

$$\left.\begin{aligned} \frac{dl_1}{ds} &= l_2\theta_3 - l_3\theta_2 \\ \frac{dl_2}{ds} &= l_3\theta_1 - l_1\theta_3 \\ \frac{dl_3}{ds} &= l_1\theta_2 - l_2\theta_1 \end{aligned}\right\} \text{......................} (1),$$

and the same equations hold if m or n be substituted for l.

Let the moving axes be the tangent, the principal normal, and the binormal.

Then, if we denote the curvature and torsion by κ and τ respectively, we have

$$\left.\begin{aligned} \theta_1 &= \tau \\ \theta_2 &= 0 \\ \theta_3 &= \kappa \end{aligned}\right\},$$

and therefore

$$\left.\begin{aligned} \frac{dl_1}{ds} &= \kappa l_2 \\ \frac{dl_2}{ds} &= \tau l_3 - \kappa l_1 \\ \frac{dl_3}{ds} &= -\tau l_2 \end{aligned}\right\} \text{......................} (2).$$

These very important formulæ we owe to Serret.

Small deformation of a curve.

Let u, v, w be the small displacements of $P(x, y, z)$ in the directions of the tangent, normal and binormal respectively.

Let the coordinates of the displaced point be ξ, η, ζ. Then

$$\left.\begin{aligned} \xi &= x + l_1u + l_2v + l_3w \\ \eta &= y + m_1u + m_2v + m_3w \\ \zeta &= z + n_1u + n_2v + n_3w \end{aligned}\right\} \text{................} (3).$$

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* Routh, *Rigid Dynamics*, II., p. 5, and Darboux, *Théorie des Surfaces*, I., p. 4.

It will be sufficient to use the first of these, the presence of three equations like the one written being always implied.

Letters applying to the new curve will be accented.

Differentiating equations (3) with respect to s, we have

$$\frac{d\xi}{ds} = \frac{dx}{ds} + l_1 \frac{du}{ds} + l_2 \frac{dv}{ds} + l_3 \frac{dw}{ds} + \frac{dl_1}{ds} u + \frac{dl_2}{ds} v + \frac{dl_3}{ds} w$$

$$= l_1 \left(1 + \frac{du}{ds} - \kappa v\right) + l_2 \left(\frac{dv}{ds} + \kappa u - \tau w\right) + l_3 \left(\frac{dw}{ds} + \tau v\right),$$

on using (2). So that

$$l_1' \frac{ds'}{ds} = l_1 \left(1 + \frac{du}{ds} - \kappa v\right) + l_2 \left(\frac{dv}{ds} + \kappa u - \tau w\right) + l_3 \left(\frac{dw}{ds} + \tau v\right).$$

Squaring and adding the three equations similar to this, and remembering that squares of u, v, w and their differential coefficients are to be neglected, we have

$$\frac{ds'}{ds} = 1 + \frac{du}{ds} - \kappa v \qquad (4).$$

So that the condition for no stretching of the curve is

$$\frac{du}{ds} - \kappa v = 0,$$

and

$$l_1' = l_1 + l_2 \left(\frac{dv}{ds} + \kappa u - \tau w\right) + l_3 \left(\frac{dw}{ds} + \tau v\right).$$

Differentiating again, and again using (2), we have

$$\begin{aligned} \kappa' l_2' \frac{ds'}{ds} = & - l_1 \kappa \left(\frac{dv}{ds} + \kappa u - \tau w\right) + l_2 \left(\kappa - \tau \frac{dw}{ds} - \tau^2 v\right) \\ & + l_3 \tau \left(\frac{dv}{ds} + \kappa u - \tau w\right) \\ & + l_2 \left(\frac{d^2 v}{ds^2} + \frac{d}{ds} \kappa u - \frac{d}{ds} \tau w\right) + l_3 \left(\frac{d^2 w}{ds^2} + \frac{d}{ds} \tau v\right). \end{aligned}$$

From which we get

$$\begin{aligned} \kappa' l_2' = \; & l_2 \left\{\kappa - \kappa \frac{du}{ds} - \tau \frac{dw}{ds} + (\kappa^2 - \tau^2) v + \frac{d^2 v}{ds^2} + \frac{d}{ds} \kappa u - \frac{d}{ds} \tau w\right\} \\ & + l_3 \left\{\tau \frac{dv}{ds} + \tau \kappa u - \tau^2 w + \frac{d^2 w}{ds^2} + \frac{d}{ds} \tau v\right\} \\ & + l_1 \left\{- \kappa \frac{dv}{ds} - \kappa^2 u + \kappa \tau w\right\}. \end{aligned}$$

Squaring and adding, we get

$$\kappa' = \kappa + u\frac{d\kappa}{ds} + \frac{d^2 v}{ds^2} + (\kappa^2 - \tau^2)\, v - \tau\frac{dw}{ds} - \frac{d}{ds}\tau w \;\ldots(5),$$

and therefore

$$l_2' = l_2 + l_3\frac{1}{\kappa}\left(\frac{d^2 w}{ds^2} + \frac{d}{ds}\tau v + \tau\frac{dv}{ds} + \tau\kappa u - \tau^2 w\right)$$
$$+ l_1\frac{1}{\kappa}\left(-\kappa\frac{dv}{ds} - \kappa^2 u + \kappa\tau w\right).$$

Having l_1' and l_2' we can at once get l_3', viz.

$$l_3' = l_3 + l_1\left(-\frac{dw}{ds} - \tau v\right) + l_2\left(-\frac{1}{\kappa}\frac{d^2 w}{ds^2} - \frac{1}{\kappa}\frac{d}{ds}\tau v - \frac{\tau}{\kappa}\frac{dv}{ds} - \tau u + \frac{\tau^2}{\kappa}w\right).$$

Differentiating again, we have, on using (3),

$$-\tau' l_2'\frac{ds'}{ds} = l_2\left\{-\tau - \kappa\frac{dw}{ds} - \kappa\tau v\right.$$
$$\left. - \frac{d}{ds}\left(\frac{1}{\kappa}\frac{d^2 w}{ds^2} + \frac{1}{\kappa}\frac{d}{ds}\tau v + \frac{\tau}{\kappa}\frac{dv}{ds} + \tau u - \frac{\tau^2}{\kappa}w\right)\right\}$$
$$+ l_3(\ldots) + l_1(\ldots);$$

therefore, on squaring and adding, we have

$$\tau' = \tau + \kappa\frac{dw}{ds} + \kappa\tau v - \tau\left(\frac{du}{ds} - \kappa v\right)$$
$$+ \frac{d}{ds}\left(\frac{1}{\kappa}\frac{d^2 w}{ds^2} + \frac{1}{\kappa}\frac{d}{ds}\tau v + \frac{\tau}{\kappa}\frac{dv}{ds} + \tau u - \frac{\tau^2}{\kappa}w\right).$$

So that, finally, we have

$$\left.\begin{aligned}
&\frac{ds'}{ds} = 1 + \frac{du}{ds} - \kappa v\\
&\kappa' - \kappa = \frac{d^2 v}{ds^2} + (\kappa^2 - \tau^2)\, v - 2\tau\frac{dw}{ds} - w\frac{d\tau}{ds} + u\frac{d\kappa}{ds}\\
&\tau' - \tau = \frac{d}{ds}\left(\frac{1}{\kappa}\frac{d^2 w}{ds^2} + 2\frac{\tau}{\kappa}\frac{dv}{ds} + \frac{v}{\kappa}\frac{d\tau}{ds} + \tau u - \frac{\tau^2}{\kappa}w\right)\\
&\qquad\qquad + \kappa\frac{dw}{ds} + 2\kappa\tau v - \tau\frac{du}{ds}
\end{aligned}\right\}\ldots(6).$$

If κ and τ are constants, we have the simpler formulæ

$$\left.\begin{aligned}
\frac{ds'}{ds} &= 1 + \frac{du}{ds} - \kappa v \\
\kappa' - \kappa &= \frac{d^2 v}{ds^2} + (\kappa^2 - \tau^2)\, v - 2\tau \frac{dw}{ds} \\
\tau' - \tau &= \frac{1}{\kappa}\left\{\frac{d^2 w}{ds^2} + (\kappa^2 - \tau^2)\frac{dw}{ds} + 2\tau \frac{d^2 v}{ds^2} + 2\kappa^2 \tau v\right\}
\end{aligned}\right\}\ldots(7).$$

These are the formulæ we shall use later on in considering the vibrations of helices and circular rings.

As Serret's formulæ do not appear in English books on Solid Geometry, it may be worth while to take another example to show the advantages of their use.

Let it be required to expand the coordinates x, y, z in powers of the arc.

We have

$$x = x_0 + \left(\frac{dx}{ds}\right)_0 s + \left(\frac{d^2x}{ds^2}\right)_0 \frac{s^2}{2} + \left(\frac{d^3x}{ds^3}\right)_0 \frac{s^3}{3\,!}$$

$$+ \left(\frac{d^4x}{ds^4}\right)_0 \frac{s^4}{4\,!} + \left(\frac{d^5x}{ds^5}\right)_0 \frac{s^5}{5\,!} + \&c.$$

$$= x_0 + l_{10} s + \left(\frac{dl_1}{ds}\right)_0 \frac{s^2}{2} + \left(\frac{d^2 l_1}{ds^2}\right)_0 \frac{s^3}{3\,!} + \left(\frac{d^3 l_1}{ds^3}\right)_0 \frac{s^4}{4\,!} + \left(\frac{d^4 l_1}{ds^4}\right)_0 \frac{s^5}{5\,!} + \&c.$$

Now

$$\frac{dl_1}{ds} = \kappa l_2,$$

$$\frac{d^2 l_1}{ds^2} = -\kappa^2 l_1 + \frac{d\kappa}{ds} l_2 + \kappa\tau l_3,$$

$$\frac{d^3 l_1}{ds^2} = -l_1 \tfrac{3}{2}\frac{d\kappa^2}{ds} + l_2\left(\frac{d^2\kappa}{ds^2} - \kappa^3 - \kappa\tau^2\right) + l_3\left(\frac{d}{ds}\kappa\tau + \tau\frac{d\kappa}{ds}\right),$$

and

$$\frac{d^4 l_1}{ds^4} = -l_1\left(\tfrac{3}{2}\frac{d^2\kappa^2}{ds^2} + \kappa\frac{d^2\kappa}{ds^2} - \kappa^4 - \kappa^2\tau^2\right)$$

$$+ l_2\left(\frac{d^3\kappa}{ds^3} - \frac{d}{ds}\kappa^3 - \frac{d}{ds}\kappa\tau^2 - \tau\frac{d}{ds}\kappa\tau - \tau^2\frac{d\kappa}{ds} - \tfrac{3}{2}\kappa\frac{d\kappa^2}{ds}\right)$$

$$+ l_3\left(\frac{d^2}{ds^2}\kappa\tau + \frac{d}{ds}\tau\frac{d\kappa}{ds} + \tau\frac{d^2\kappa}{ds^2} - \kappa^3\tau - \kappa\tau^3\right).$$

Now let the axes x, y, z be the tangent, principal normal, and binormal at x_0, y_0, z_0.

We at once deduce

$$x = s - \kappa^2 \frac{s^3}{3\,!} - \tfrac{3}{2} \frac{d\kappa^2}{ds} \frac{s^4}{4\,!} - \left(\tfrac{3}{2} \frac{d^2\kappa^2}{ds^2} + \kappa \frac{d^2\kappa}{ds^2} - \kappa^4 - \kappa^2\tau^2 \right) \frac{s^5}{5\,!} + \dots,$$

$$y = \kappa \frac{s^2}{2\,!} + \frac{d\kappa}{ds} \frac{s^3}{3\,!} + \left(\frac{d^2\kappa}{ds^2} - \kappa^3 - \kappa\tau^2 \right) \frac{s^4}{4\,!}$$

$$+ \left(\frac{d^3\kappa}{ds^3} - \frac{d}{ds}\kappa^3 - \frac{d}{ds}\kappa\tau^2 - \tau \frac{d}{ds}\kappa\tau - \tau^2 \frac{d\kappa}{ds} - \tfrac{3}{2}\kappa \frac{d\kappa^2}{ds} \right) \frac{s^5}{5\,!} + \dots,$$

$$z = \kappa\tau \frac{s^3}{3\,!} + \left(\frac{d}{ds}\kappa\tau + \tau \frac{d\kappa}{ds} \right) \frac{s^4}{4\,!}$$

$$+ \left(\frac{d^2}{ds^2}\kappa\tau + \frac{d}{ds}\tau \frac{d\kappa}{ds} + \tau \frac{d^2\kappa}{ds^2} - \kappa^3\tau - \kappa\tau^3 \right) \frac{s^5}{5\,!} + \dots.$$

Small deformation of a surface.

The surface is supposed striated by a double set of curves whose parameters are p, q cutting orthogonally.

The moving axes are taken to be the normal to the surface and the tangents to the two curves p, q intersecting at the point.

Let (x, y, z) be the coordinates of the point $P(p, q)$ on the surface referred to fixed axes, and let the direction cosines of the axes at P be $(l_1m_1n_1)$, $(l_2m_3n_2)$, $(l_3m_3n_3)$, the last being the normal to the surface.

Let κ_1, κ_2 be the normal curvatures of p, q at $P(xyz)$ and λ_1, λ_2 the curvatures in the tangent plane.

Let H be the geodesic torsion along $(l_1m_1n_1)$, and therefore $-H$ the geodesic torsion along $(l_2m_2n_2)$.

Then the rates of twist of the moving axes as we go along q constant are H, κ_1, λ_1 about the three axes respectively, and as we go along p constant the rates of twist are $-\kappa_2$, $-H, \lambda_2$ about the same axes, where the signs are so chosen that the normal curvatures are both measured from the normal and the tangential curvatures in the same direction round the normal.

The name *geodesic torsion* was given by Bonnet, to whom, with Bertrand, we owe the notion of H as a twist (Darboux, *Theorie des Surfaces*, II., p. 388).

It is easy to show that H is the coefficient of the middle term in Euler's formula for the normal curvature, viz. in a direction making an angle θ with $(l_1 m_1 n_1)$, we have for the curvature $\frac{1}{R}$

$$\frac{1}{R} = \kappa_1 \cos^2\theta + 2H\cos\theta\sin\theta + \kappa_2 \sin^2\theta.$$

Now let elements of arc along q constant, p constant, be

$$ds_1 = a\,dp, \quad ds_2 = b\,dq.$$

Then from the fundamental twist formulæ we have

$$\left.\begin{aligned}
\frac{dl_1}{ds_1} &= \quad l_2\lambda_1 - l_3\kappa_1 \\
\frac{dl_2}{ds_1} &= \quad l_3 H - l_1\lambda_1 \\
\frac{dl_3}{ds_1} &= \quad l_1\kappa_1 - l_2 H \\
\frac{dl_1}{ds_2} &= \quad l_2\lambda_2 + l_3 H \\
\frac{dl_2}{ds_2} &= -l_3\kappa_2 - l_1\lambda_2 \\
\frac{dl_3}{ds_2} &= -l_1 H + l_2\kappa_2
\end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (8).$$

Now let the point $P(xyz)$ receive small displacements u, v, w in the directions of the axes at P, and let (ξ, η, ζ) be the new coordinates of P.

Then
$$\left.\begin{aligned}
\xi &= x + l_1 u + l_2 v + l_3 w \\
\eta &= y + m_1 u + m_2 v + m_3 w \\
\zeta &= z + n_1 u + n_2 v + n_3 w
\end{aligned}\right\}.$$

As before we shall write only one equation, the other two being implied.

Differentiating the first with respect to s_1, we have

$$l'_1 \frac{ds'_1}{ds_1} = l_1\left(1 + \frac{du}{ds_1} - \lambda_1 v + \kappa_1 w\right) + l_2\left(\frac{dv}{ds_1} - Hw + \lambda_1 u\right)$$
$$+ l_3\left(\frac{dw}{ds_1} - \kappa_1 u + Hv\right).$$

Squaring and adding, we have

$$\frac{ds'_1}{ds_1} = 1 + \frac{du}{ds_1} - \lambda_1 v + \kappa_1 w,$$

and $$l_1' = l_1 + l_2\left(\frac{dv}{ds_1} - Hw + \lambda_1 u\right) + l_3\left(\frac{dw}{ds_1} - \kappa_1 u + Hv\right).$$

Similarly $$\frac{ds'_2}{ds_2}=1+\frac{dv}{ds_2}+\kappa_2 w+\lambda_2 u,$$

and $$l_2'=l_2+l_3\left(\frac{dw}{ds_2}+Hu-\kappa_2 v\right)+l_1\left(\frac{du}{ds_2}-\lambda_2 v-Hw\right).$$

Let θ be the angle between the two tangents after displacement.

Then $$\cos\theta=\frac{du}{ds_2}-\lambda_2 v-Hw+\frac{dv}{ds_1}-Hw+\lambda_1 u$$
$$=\frac{du}{ds_2}+\frac{dv}{ds_1}+\lambda_1 u-\lambda_2 v-2Hw.$$

Thus the conditions for no stretching of the surface are

$$\left.\begin{array}{l}\dfrac{du}{ds_1} \qquad -\lambda_1 v+\kappa_1 w \qquad\qquad =0\\ \qquad \dfrac{dv}{ds_2}+\kappa_2 w+\lambda_2 u \qquad\quad =0\\ \dfrac{du}{ds_2}+\dfrac{dv}{ds_1}+\lambda_1 u-\lambda_2 v-\kappa H w=0\end{array}\right\}\text{............(9).}$$

Now let $(l_3'm'_3n'_3)$ be the new normal, and $(l_4m_4n_4)$, $(l_5m_5n_5)$ perpendicular to $(l_1'm'_1n'_1)$, $(l_2'm'_2n'_2)$ respectively, and in the tangent plane.

These are deduced immediately from l_1' and l_2', viz.

$$\left.\begin{array}{l}l_3'=l_3-l_1\left(\dfrac{dw}{ds_1}-\kappa_1 u+Hv\right)-l_2\left(\dfrac{dw}{ds_2}+Hu-\kappa_2 v\right)\\ l_4=l_2+l_3\left(\dfrac{dw}{ds_2}+Hu-\kappa_2 v\right)-l_1\left(\dfrac{dv}{ds_1}-Hw+\lambda_1 u\right)\\ l_5=l_1-l_2\left(\dfrac{du}{ds_2}-\lambda_2 v-Hw\right)+l_3\left(\dfrac{dw}{ds_1}-\kappa_1 u+Hv\right)\end{array}\right\}\text{...(10).}$$

Differentiate l_1' with respect to s_1 and we get

$$[l_4\lambda_1'-l_3'\kappa'_1]\frac{ds_1'}{ds_1}$$
$$=\ l_1\left(\kappa_1\frac{dw}{ds_1}-\kappa_1^2 u+\kappa_1 Hv-\lambda_1\frac{dv}{ds_1}+\lambda_1 Hw-\lambda_1^2 u\right)$$
$$+l_2\left(\lambda_1+\frac{d^2v}{ds_1^2}-\frac{d}{ds_1}Hw+\frac{d}{ds_1}\lambda_1 u-H\frac{dw}{ds_1}+H\kappa_1 u-H^2 v\right)$$
$$+l_3\left(-\kappa_1+\frac{d^2w}{ds_1^2}-\frac{d}{ds_1}\kappa_1 u+\frac{d}{ds_1}Hv+H\frac{dv}{ds_1}-H^2 w+H\lambda_1 u\right).$$

Now multiply by l_3', m_3', n' respectively, and add.

We thus get

$$-\kappa'_1 \frac{ds'_1}{ds_1} = -\lambda_1 \left(\frac{dw}{ds_2} + Hu - \kappa_2 v\right)$$

$$+ \left(-\kappa_1 + \frac{d^2 w}{ds_1^2} - \frac{d}{ds_1} \kappa_1 u + \frac{d}{ds_1} Hv + H\frac{dv}{ds_1} - H^2 w + H\lambda_1 u\right),$$

and therefore

$$\kappa'_1 - \kappa_1 = -\frac{d^2 w}{ds_1^2} + u\frac{d\kappa_1}{ds_1} - \frac{d}{ds_1} Hv - H\frac{dv}{ds_1} + H^2 w$$

$$+ \lambda_1 \frac{dw}{ds_2} - \kappa_2 \lambda_1 v + \kappa_1 \lambda_1 v - \kappa_1^2 w.$$

Multiply the same equations by $l_4 m_4 n_4$ and add, and we get

$$\lambda'_1 \frac{ds_1'}{ds_1} = \lambda_1 + \frac{d^2 v}{ds_1^2} - \frac{d}{ds_1} Hw + \frac{d}{ds_1} \lambda_1 u - H\frac{dw}{ds_1} + H\kappa_1 u - H^2 v$$

$$- \kappa_1 \left(\frac{dw}{ds_2} + Hu - \kappa_2 v\right);$$

therefore

$$\lambda'_1 - \lambda_1 = \frac{d^2 v}{ds_1^2} - \frac{d}{ds_1} Hw + u\frac{d\lambda_1}{ds_1} - H\frac{dw}{ds_1} - H^2 v$$

$$- \kappa_1 \frac{dw}{ds_2} + \kappa_1 \kappa_2 v + \lambda_1^2 v - \lambda_1 \kappa_1 w.$$

In like manner we have

$$\kappa'_2 - \kappa_2 = -\frac{d^2 w}{ds_2^2} + v\frac{d\kappa_2}{ds_2} - \frac{d}{ds_2} Hu - H\frac{du}{ds_2} + H^2 w$$

$$- \lambda_2 \frac{dw}{ds_1} + \kappa_1 \lambda_2 u - \kappa_2 \lambda_2 u - \kappa_2^2 w,$$

$$\lambda'_2 - \lambda_2 = \frac{d^2 u}{ds_2^2} - \frac{d}{ds_2} Hw - v\frac{d\lambda_2}{ds_2} - H\frac{dw}{ds_2} - H^2 u$$

$$- \kappa_2 \frac{dw}{ds_1} + \kappa_1 \kappa_2 u + \lambda_2^2 u + \kappa_2 \lambda_2 w.$$

If now we differentiate l_4 with respect to s_1 we get

$$(l'_3 H_1' - l_1'\lambda_1') \frac{ds'_1}{ds_1} = l_1 \left(-\lambda_1 - \frac{d^2 v}{ds_1^2} + \frac{d}{ds_1} Hw - \frac{d}{ds_1} \lambda_1 u\right)$$

$$+ l_2 \left(-H\frac{dw}{ds_2} - H^2 u + H\kappa^2 v - \lambda_1 \frac{dv}{ds_1} + \lambda_1 Hw - \lambda_1^2 u\right)$$

$$+ l_3 \left(H + \frac{d^2 w}{ds_1 ds_2} + \frac{d}{ds_1} Hu - \frac{d}{ds_1} \kappa_2 v + \kappa_1 \frac{dv}{ds_1} - \kappa_1 Hw + \kappa_1 \lambda_1 u\right).$$

Multiply by l_3', m_3', n_3' respectively and add, giving

$$H_1'\frac{ds_1'}{ds_1}=H+\frac{d^2w}{ds_1\,ds_2}+\frac{d}{ds_1}Hu-\frac{d}{ds_1}\kappa_2 v+\kappa_1\frac{dv}{ds_1}-\kappa_1 Hw$$
$$+\kappa_1\lambda_1 u+\lambda_1\left(\frac{dw}{ds_1}-\kappa_1 u+Hv\right);$$

therefore

$$H_1'-H=\frac{d^2w}{ds_1\,ds_2}-H\left(\frac{du}{ds_1}-\lambda_1 v+\kappa_1 w\right)+\frac{d}{ds_1}Hu-\frac{d}{ds_1}\kappa_2 v$$
$$+\kappa_1\frac{dv}{ds_1}-\kappa_1 Hw+\kappa_1\lambda_1 u+\lambda_1\left(\frac{dw}{ds_1}-\kappa_1 u+Hv\right).$$

Similarly

$$H_2'-H=\frac{d^2w}{ds_2\,ds_1}-H\left(\frac{dv}{ds_2}+\lambda_2 u+\kappa_2 w\right)+\frac{d}{ds_2}Hv-\frac{d}{ds_2}\kappa_1 u$$
$$+\kappa_2\frac{du}{ds_2}-\kappa_2 Hw-\kappa_2\lambda_2 v-\lambda_2\left(\frac{dw}{ds_2}-\kappa_2 v+Hu\right).$$

The two most important cases are (1) when the lines p, q are lines of curvature, (2) when the lines q are geodesic radii.

For the first case* we have

$$\left.\begin{aligned}\frac{ds_1'}{ds_1}&=1+\frac{du}{ds_1}-\lambda_1 v+\kappa_1 w\\ \frac{ds_2'}{ds_2}&=1+\frac{dv}{ds_2}+\lambda_2 u+\kappa_2 w\\ \cos\theta&=\frac{du}{ds_2}+\frac{dv}{ds_1}+\lambda_1 u-\lambda_2 v\end{aligned}\right\},$$

$$\left.\begin{aligned}\kappa_1'-\kappa_1&=-\frac{d^2w}{ds_1^2}+\lambda_1\frac{dw}{ds_2}-\kappa_1^2 w+u\frac{d\kappa_1}{ds_1}+\lambda_1(\kappa_1-\kappa_2)v\\ \kappa_2'-\kappa_2&=-\frac{d^2w}{ds_2^2}-\lambda_2\frac{dw}{ds_1}-\kappa_2^2 w+v\frac{d\kappa_2}{ds_2}-\lambda_2(\kappa_2-\kappa_1)u\\ H_1'-H&=\frac{d^2w}{ds_1\,ds_2}+\lambda_1\frac{dw}{ds_1}+(\kappa_1-\kappa_2)\frac{dv}{ds_1}-v\frac{d\kappa_2}{ds_1}\\ H_2'-H&=\frac{d^2w}{ds_2\,ds_1}-\lambda_2\frac{dw}{ds_2}+(\kappa_2-\kappa_1)\frac{du}{ds_2}-u\frac{d\kappa_1}{ds_2}\end{aligned}\right\},$$

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* cf. Love, *Phil. Trans.*, 1888, pp. 506 and 509.

For the second case

$$\left.\begin{aligned} \frac{ds_1'}{ds_1} &= 1 + \frac{du}{ds_1} + \kappa_1 w \\ \frac{ds_2'}{ds_2} &= 1 + \frac{dv}{ds_2} + \lambda_2 u + \kappa_2 w \\ \cos\theta &= \frac{du}{ds_2} + \frac{dv}{ds_1} - \lambda_2 v - 2Hw \end{aligned}\right\};$$

$$\left.\begin{aligned} \kappa'_1 - \kappa_1 &= -\frac{d^2w}{ds_1^2} + (H^2 - \kappa_1^2)\,w + u\frac{d\kappa_1}{ds_1} - \frac{d}{ds_1} Hv - H\frac{dv}{ds_1} \\ \kappa'_2 - \kappa_2 &= -\frac{d^2w}{ds_2^2} - \lambda_2 \frac{dw}{ds_1} + (H^2 - \kappa_2^2)\,w + \lambda_2(\kappa_1 - \kappa_2)\,u \\ &\quad - \frac{d}{ds_2} Hu - H\frac{du}{ds_2} + v\frac{d\kappa_2}{ds_2} \\ H_1' - H &= \frac{d^2w}{ds_1\,ds_2} - 2\kappa_1 Hw + u\frac{dH}{ds_1} - v\frac{d\kappa_2}{ds_1} + (\kappa_1 - \kappa_2)\frac{dv}{ds_1} \\ H_2' - H &= \frac{d^2w}{ds_2\,ds_1} - \lambda_2\frac{dw}{ds_2} - 2\kappa_2 Hw - u\frac{d\kappa_1}{ds_2} + (\kappa_2 - \kappa_1)\frac{du}{ds_2} \\ &\quad - 2H\lambda_2 u + v\frac{dH}{ds_2} \end{aligned}\right\}.$$

If we make $u = v = 0$, the condition that the lines p, q may be lines of curvature on the new surface is

$$\frac{d^2w}{ds_1\,ds_2} + \lambda_1\frac{dw}{ds_1} = \frac{d^2w}{ds_2\,ds_1} - \lambda_2\frac{dw}{ds_2} = 0;$$

that this is but one equation is seen by putting

$$ds_1 = a\,dp,$$

$$ds_2 = b\,dq,$$

and remembering that

$$\lambda_1 = -\frac{1}{ab}\frac{da}{dq},$$

$$\lambda_2 = \frac{1}{ab}\frac{db}{dp},$$

so that either equation reduces to

$$\frac{d^2w}{dp\,dq} - \frac{1}{b}\frac{db}{dp}\frac{dw}{dq} - \frac{1}{a}\frac{da}{dq}\frac{dw}{dp} = 0;$$

this is equivalent to the equation first given by Cayley as the condition that two adjacent surfaces should belong to the same orthogonal system (Salmon, *Geometry of Three Dimensions*, § 478).

Vibrations of a wire.

Following Thomson and Tait we define the axis of the wire as the locus of the centres of inertia of its sections, and a transverse as a line fixed in the wire passing through the axis and normal to it.

We shall take the transverse in the unstrained state of the wire to coincide with the principal normal at the point.

Let α be the inclination of the transverse to the osculating plane after deformation.

Then the curvatures along the transverse and perpendicular to it are $\kappa' \cos\alpha$, $\kappa' \sin\alpha$.

The twist of the wire is

$$\frac{d\alpha}{ds} + \tau' - \tau.$$

If then one of the principal axes of elasticity coincide with the tangent line and the moduli of the other two are equal, the potential energy per unit length is

$$\tfrac{1}{2}L\left\{(\kappa'\cos\alpha - \kappa)^2 + \kappa'^2\sin^2\alpha\right\} + \tfrac{1}{2}M\left\{\frac{d\alpha}{ds} + \tau' - \tau\right\}^2$$

$$= \tfrac{1}{2}L\left\{(\kappa' - \kappa)^2 + \kappa^2\alpha^2\right\} + \tfrac{1}{2}M\left\{\frac{d\alpha}{ds} + \tau' - \tau\right\}^2$$

neglecting powers of small quantities above the second.

The vibrations to be considered are such that there is no stretching of the wire. This gives the condition

$$\frac{du}{ds} - \kappa v = 0.$$

If we substitute for v from this equation in the expressions for $\kappa' - \kappa$, $\tau' - \tau$, we can express the potential energy in terms of u, w, and α, and the virtual moments of the acceleration can be expressed in terms of the same quantities.

The use of Lagrange's method then leads to the general equations of motion.

We shall apply this to the case of a circular helix.

In this case

$$\kappa' - \kappa = \frac{1}{\kappa}\frac{d^3u}{ds^3} + \frac{1}{\kappa}(\kappa^2 - \tau^2)\frac{du}{ds} - 2\tau\frac{dw}{ds},$$

$$\tau' - \tau = \frac{1}{\kappa}\frac{d^3w}{ds^3} + \frac{1}{\kappa}(\kappa^2 - \tau^2)\frac{dw}{ds} + \frac{2\tau}{\kappa^2}\frac{d^3u}{ds^3} + 2\tau\frac{du}{ds}.$$

Calling V the potential energy of the wire, we have, for the variation of this,

$$\delta V = \int ds\, L\left\{\frac{1}{\kappa}\frac{d^3u}{ds^3} + \frac{1}{\kappa}(\kappa^2 - \tau^2)\frac{du}{ds} - 2\tau\frac{dw}{ds}\right\}$$

$$\times\left\{\frac{1}{\kappa}\frac{d^3\delta u}{ds^3} + \frac{1}{\kappa}(\kappa^2 - \tau^2)\frac{d\delta u}{ds} - 2\tau\frac{d\delta w}{ds}\right\}$$

$$+ L\kappa^2\alpha\delta\alpha + M\left[\left\{\frac{d\alpha}{ds} + \frac{2\tau}{\kappa^2}\frac{d^3u}{ds^3} + 2\tau\frac{du}{ds} + \frac{1}{\kappa}\frac{d^3w}{ds^3} + \frac{1}{\kappa}(\kappa^2 - \tau^2)\frac{dw}{ds}\right\}\right.$$

$$\left.\times\left\{\frac{d\delta\alpha}{ds} + \frac{2\tau}{\kappa^2}\frac{d^3\delta u}{ds^3} + 2\tau\frac{d\delta u}{ds} + \frac{1}{\kappa}\frac{d^3\delta w}{ds^3} + \frac{1}{\kappa}(\kappa^2 - \tau^2)\frac{d\delta w}{ds}\right\}\right].$$

If we integrate by parts, the portion remaining under the integral sign is, writing $\kappa^2 - \tau^2 = \sigma^2$,

$$\int ds\,\delta u\left[-L\left(\frac{1}{\kappa}\frac{d^3}{ds^3} + \frac{\sigma^2}{\kappa}\frac{d}{ds}\right)\left(\frac{1}{\kappa}\frac{d^3u}{ds^3} + \frac{\sigma^2}{\kappa}\frac{du}{ds} - 2\tau\frac{dw}{ds}\right)\right.$$

$$\left.- M\left(\frac{2\tau}{\kappa^2}\frac{d^3}{ds^3} + 2\tau\frac{d}{ds}\right)\left(\frac{d\alpha}{ds} + \frac{2\tau}{\kappa^2}\frac{d^3u}{ds^3} + 2\tau\frac{du}{ds} + \frac{1}{\kappa}\frac{d^3w}{ds^3} + \frac{\sigma^2}{\kappa}\frac{dw}{ds}\right)\right]$$

$$+ \delta w\left[L\,2\tau\frac{d}{ds}\left(\frac{1}{\kappa}\frac{d^3u}{ds^3} + \frac{\sigma^2}{\kappa}\frac{du}{ds} - 2\tau\frac{dw}{ds}\right)\right.$$

$$\left.- M\left(\frac{1}{\kappa}\frac{d^3}{ds^3} + \frac{\sigma^2}{\kappa}\frac{d}{ds}\right)\left(\frac{d\alpha}{ds} + \frac{2\tau}{\kappa^2}\frac{d^3u}{ds^3} + 2\tau\frac{du}{ds} + \frac{1}{\kappa}\frac{d^3w}{ds^3} + \frac{\sigma^2}{\kappa}\frac{dw}{ds}\right)\right]$$

$$+ \delta\alpha\left[L\kappa^2\alpha - M\frac{d}{ds}\left\{\frac{d\alpha}{ds} + \frac{2\tau}{\kappa^2}\frac{d^3u}{ds^3} + 2\tau\frac{du}{ds} + \frac{1}{\kappa}\frac{d^3w}{ds^3} + \frac{\sigma^2}{\kappa}\frac{dw}{ds}\right\}\right].$$

The part of the virtual moments due to acceleration is

$$m\int(\ddot{u}\,\delta u + \ddot{v}\,\delta v + \ddot{w}\,\delta w)\,ds.$$

Now

$$\int\ddot{v}\,\delta v\,ds = \frac{1}{\kappa^2}\int\frac{d^3u}{ds\,dt^2}\frac{d\delta u}{ds}\,ds = \frac{1}{\kappa^2}\left[\frac{d^3u}{ds\,dt^2}\,\delta u\right] - \frac{1}{\kappa^2}\int\frac{d^4u}{ds^2\,dt^2}\,\delta u\,ds.$$

The unintegrated part is therefore

$$m\int\left[\left(\ddot{u}-\frac{1}{\kappa^2}\frac{d^4u}{ds^2dt^2}\right)\delta u+\ddot{w}\,\delta w\right]ds.$$

From this and the value of δV we can at once write down the equations of motion.

They are

$$m\left(\frac{d^2u}{dt^2}-\frac{1}{\kappa^2}\frac{d^4u}{ds^2dt^2}\right)=L\left(\frac{1}{\kappa}\frac{d^3}{ds^3}+\frac{\sigma^2}{\kappa}\frac{d}{ds}\right)\left(\frac{1}{\kappa}\frac{d^3u}{ds^3}+\frac{\sigma^2}{\kappa}\frac{du}{ds}-2\tau\frac{dw}{ds}\right)$$

$$+M\left(\frac{2\tau}{\kappa^2}\frac{d^3}{ds^3}+2\tau\frac{d}{ds}\right)\left(\frac{d\alpha}{ds}+\frac{2\tau}{\kappa^2}\frac{d^3u}{ds^3}+2\tau\frac{du}{ds}+\frac{1}{\kappa}\frac{d^3w}{ds^3}+\frac{\sigma^2}{\kappa}\frac{dw}{ds}\right),$$

$$m\frac{d^2w}{dt^2}=-L2\tau\frac{d}{ds}\left(\frac{1}{\kappa}\frac{d^3u}{ds^3}+\frac{\sigma^2}{\kappa}\frac{du}{ds}-2\tau\frac{dw}{ds}\right)$$

$$+M\left(\frac{1}{\kappa}\frac{d^3}{ds^3}+\frac{\sigma^2}{\kappa}\frac{d}{ds}\right)\left(\frac{d\alpha}{ds}+\frac{2\tau}{\kappa^2}\frac{d^3u}{ds^3}+2\tau\frac{du}{ds}+\frac{1}{\kappa}\frac{d^3w}{ds^3}+\frac{\sigma^2}{\kappa}\frac{dw}{ds}\right),$$

$$0=L\kappa^2\alpha-M\frac{d}{ds}\left\{\frac{d\alpha}{ds}+2\frac{\tau}{\kappa^2}\frac{d^3u}{ds^3}+2\tau\frac{du}{ds}+\frac{1}{\kappa}\frac{d^3w}{ds^3}+\frac{\sigma^2}{\kappa}\frac{dw}{ds}\right\}.$$

If we put

$$u=Ae^{i(pt-ns)}\;w=Be^{i(pt-ns)}\;\alpha=Ce^{i(pt-ns)},$$

and, after substituting from the last equation in the first two eliminate A, B, C, we get the determinantal equation

$$\begin{vmatrix} mp^2\left(1+\frac{n^2}{\kappa^2}\right)-Ln^2\left(\frac{\sigma^2}{\kappa}-\frac{n^2}{\kappa}\right)^2, & 2Ln^2\tau\left(\frac{\sigma^2}{\kappa}-\frac{n^2}{\kappa}\right), & L\kappa^2\left(2\tau-\frac{2\tau}{\kappa^2}n^2\right) \\ 2L\tau n^2\left(\frac{\sigma^2}{\kappa}-\frac{n^2}{\kappa}\right), & mp^2-4L\tau^2n^2, & L\kappa^2\left(\frac{\sigma^2}{\kappa}-\frac{n^2}{\kappa}\right) \\ Mn^2\left(2\tau-\frac{2\tau}{\kappa^2}n^2\right), & Mn^2\left(\frac{\sigma^2}{\kappa}-\frac{n^2}{\kappa}\right), & L\kappa^2+Mn^2 \end{vmatrix}=0,$$

from which we can see that there are two kinds of waves of given length which travel with different velocities.

When the helix is of small pitch the natures of the two are very different. For, if we neglect τ, the equations separate into the two sets

$$m\left(\frac{d^2u}{dt^2}-\frac{1}{\kappa^2}\frac{d^4u}{ds^2dt^2}\right)=L\left(\frac{1}{\kappa}\frac{d^3}{ds^3}+\kappa\frac{d}{ds}\right)^2u,$$

and

$$\left.\begin{aligned} m\frac{d^2w}{dt^2} &= L\left(\kappa\frac{d^2\alpha}{ds^2}+\kappa^3\alpha\right) \\ 0 &= L\kappa^2\alpha - M\left(\frac{d^2\alpha}{ds^2}+\frac{1}{\kappa}\frac{d^4w}{ds^4}+\kappa\frac{d^2w}{ds^2}\right) \end{aligned}\right\}.$$

Consequently there will be one wave which will appear as a progressive contraction of the radius of the helix, and whose velocity will be given by

$$\frac{p}{n}=\sqrt{\left(\frac{L}{m}\right)}\frac{\kappa^2-n^2}{\sqrt{(\kappa^2+n^2)}},$$

and another wave which will appear as a crowding of the successive turns of the helix, and whose velocity will be given by

$$\frac{p}{n}=\sqrt{\left(\frac{ML}{m}\right)}\frac{\kappa^2-n^2}{\sqrt{(L\kappa^2+Mn^2)}}.$$

Vibrations of a circular ring.

The equations we have written down for the case $\tau=0$ are clearly the equations of vibration of a circular ring.

In order to adapt the above formulæ to this case it is only necessary to notice that here n must be of the form $r\kappa$ where r is an integer, and $a\kappa=1$ where a is the radius. Thus, for vibrations in the plane of the ring, we have

$$p^2=\frac{L}{m}\frac{1}{a^4}\frac{(1-r^2)^2}{1+r^2},$$

and for vibrations normal to the plane of the ring, and involving torsion,

$$p^2=\frac{ML}{m}\frac{1}{a^4}\frac{(1-r^2)^2}{L+Mr^2}.$$

The first of these results was given by Hoppe in *Crelle*, Bd. 63, 1871.

ON THE EXHAUSTION OF NEUMANN'S MODE OF SOLUTION FOR THE MOTION OF SOLIDS OF REVOLUTION IN LIQUIDS, AND SIMILAR PROBLEMS.

By *J. H. Michell, B.A.*, Scholar of Trinity College, Cambridge.

The method in question is explained in Basset's *Hydrodynamics*, vol. II., Art. 254.

Laplace's equation when transformed into cylindrical coordinates z, ϖ, and ϕ becomes

$$\frac{d^2V}{dz^2}+\frac{d^2V}{d\varpi^2}+\frac{1}{\varpi}\frac{dV}{d\varpi}+\frac{1}{\varpi^2}\frac{d^2V}{d\phi^2}=0.$$

If we put

$$V=U\varpi^{-\frac{1}{2}}\sin(m\phi+\alpha),$$

we get

$$\frac{d^2U}{dz^2}+\frac{d^2U}{d\varpi^2}+\frac{1}{\varpi^2}\left(\tfrac{1}{4}-m^2\right)U=0.$$

Let $$z+i\varpi=f(\xi+i\eta),$$

and $$J^2=\left(\frac{d\xi}{dz}\right)^2+\left(\frac{d\xi}{d\varpi}\right)^2,$$

then, on transforming, we get

$$\frac{d^2U}{d\xi^2}+\frac{d^2U}{d\eta^2}+\frac{1}{J^2\varpi^2}\left(\tfrac{1}{4}-m^2\right)U=0.$$

In order that this may admit of a solution in the form $U=\Sigma\,\phi(\xi)\,\chi(\eta)$ it is necessary and sufficient that $\frac{1}{J^2\varpi^2}$ should be the sum of a function of ξ and a function of η.

The object of the present note is to find all the transformations

$$z+i\varpi=f(\xi+i\eta),$$

for which this condition is satisfied.

Since $$\frac{1}{J^2}=\frac{\partial(z+i\varpi)}{\partial(\xi+i\eta)}\,\frac{\partial(z-i\varpi)}{\partial(\xi-i\eta)},$$

we have $$\frac{f'(\xi+i\eta)\,f'(\xi-i\eta)}{[f(\xi+i\eta)-f(\xi-i\eta)]^2}=\phi(\xi)+\psi(\eta),$$

where $\phi(\xi)$, $\psi(\eta)$ are arbitrary functions.

If we denote the left-hand side by P, we have

$$\frac{d^2}{d\xi\, d\eta}(P)=0.$$

Now put
$$f(\xi+i\eta)=u,$$
$$f(\xi-i\eta)=v,$$
so that
$$f'(\xi+i\eta)=u',$$
and so on, and we have

$$\frac{d^2}{d\xi\, d\eta}\,\frac{u'v'}{(u-v)^2}=0;$$

therefore
$$\frac{d}{d\eta}\left[\frac{u''v'+u'v''}{(u-v)^3}-2\,\frac{u'v'\,(u'-v')}{(u-v)^3}\right]=0,$$

and therefore

$$\begin{aligned}&(u'''v'-u'v''')\,(u-v)^2\\ &\qquad -4\,(u-v)\,\{(u''v'+u'v'')\,(u'+v')-(u''v'-u'v'')\,(u'-v')\}\\ &\qquad +6\,(u'+v')\,(u'-v')\,u'v'-2u'v'\,(u''+v'')\,(u-v)=0,\end{aligned}$$

which reduces to

$$\begin{aligned}&\frac{1}{u'}\,(u'''u^2+6u'^3-6uu'u'')+2v\left(-\frac{uu'''}{u'}+3u''\right)+v^2\,\frac{u'''}{u'}\\ &=\frac{1}{v'}\,(v'''v^2+6v'^3-6vv'v'')+2u\left(-\frac{vv'''}{v'}+3v''\right)+u^2\,\frac{v'''}{v'}.\end{aligned}$$

Since u and v are independent, we can satisfy this only by putting

$$\begin{aligned}\frac{1}{u'}\,(u'''u^2+6u'^3-6uu'u'')&=A+Bu+Cu^2\\ -\frac{uu'''}{u'}+3u''&=D+Eu+Fu^2,\\ \frac{u'''}{u'}&=G+Hu+Ku^2,\end{aligned}$$

and v will satisfy the same equations.

On making this substitution, we get

$$H=2F,$$
$$G=C,$$
$$2D=B,$$

so that
$$\frac{1}{u'}(u'''u^2+6u'^3-6uu'u'')=A+Bu+Cu^2$$
$$-\frac{uu'''}{u'}+3u''=\tfrac{1}{2}B+Eu+Fu^2,$$
$$\frac{u'''}{u'}=C+2Fu+Ku^2.$$

It remains to be seen whether these can be simultaneously satisfied.

The last gives
$$u''=L+Cu+Fu^2+\tfrac{1}{3}Ku^3 \ldots\ldots\ldots\ldots\ldots (1).$$

Substituting in the second, we get
$$-u(C+2Fu+Ku^2)+3(L+Cu+Fu^2+\tfrac{1}{3}Ku^3)=\tfrac{1}{2}B+Eu+Fu^2,$$
which is satisfied if $E=2C$, $B=6L$.

The equation (1) gives
$$u'^2=M+2Lu+Cu^2+\tfrac{2}{3}Fu^3+\tfrac{1}{6}Ku^4.$$

Hence, substituting in the first of the three equations, we have
$$u^2(C+2Fu+Ku^2)+6M+12Lu+6Cu^2+4Fu^3+Ku^4$$
$$-(6Lu+6Cu^2+6Fu^3+2Ku^4)=A+Bu+Cu^2,$$
and this is satisfied if
$$B=6L,\ A=6M.$$

Therefore $\quad u'^2=\alpha+\beta u+\gamma u^2+\delta u^3+\varepsilon u^4$

satisfies the equation with which we started, there being no relation between the coefficients. Thus the most general transformation to which Neumann's method is applicable is
$$\xi+i\eta=\int\frac{du}{\sqrt{(\alpha+\beta u+\gamma u^2+\delta u^3+\varepsilon u^4)}}.$$

All the cases of this transformation which do not involve elliptic functions have been noticed and discussed; most, if not all, are treated by Basset *loc. cit.*

Wangerin seems to have first discovered that the elliptic functions sn, cn, dn, are cases included, but the ordinary differential equations to which the method leads in these cases are so complicated that there seems to be little utility in considering them further in the present state of mathematics.

In the problem of the vibration of membranes there is a like exhaustion.

The equation of motion

$$\frac{d^2\omega}{dx^2}+\frac{d^2\omega}{dy^2}=k^2\frac{d^2\omega}{dt^2},$$

on assuming a time factor e^{ipt} becomes

$$\frac{d^2\omega}{dx^2}+\frac{d^2\omega}{dy^2}+k^2p^2\omega=0.$$

On transforming to (ξ, η) where

$$x+iy=f(\xi+i\eta),$$

we get $$\frac{d^2\omega}{d\xi^2}+\frac{d^2\omega}{d\eta^2}+k^2p^2\left\{\left(\frac{dx}{d\xi}\right)^2+\left(\frac{dx}{d\eta}\right)^2\right\}\omega.$$

This can be solved in the form

$$\omega=\Sigma\,\phi(\xi)\,\chi(\eta),$$

if $$\left(\frac{dx}{d\xi}\right)^2+\left(\frac{dx}{d\eta}\right)^2=F(\xi)+G(\eta),$$

where F, G are arbitrary.

If then $$x+iy=u,\quad x-iy=v,$$

we have $$u'v'=F(\xi)+G(\eta),$$

therefore $$\frac{d^2}{d\xi\,d\eta}u'v'=0,$$

whence $$\frac{d}{d\eta}(u''v'+u'v'')=0,$$

so that $$u'''v'-u'v'''=0,$$

or $$\frac{u'''}{u'}=\frac{v'''}{v'},$$

therefore $$\frac{u'''}{u'}=\text{a const.}=k,$$

integrating $$u'=A\cosh\{\sqrt{k}\,(\xi+i\eta)+\alpha\},$$

so that $$x+iy=B\sinh\{\sqrt{k}\,(\xi+i\eta)+\alpha\}.$$

The elliptic transformation is thus the only one satisfying the condition.

The solution for the circular membrane is well known. That for the elliptic membrane was discussed by Mathieu in *Liouville's Journal*, 1868.

VIBRATIONS OF A STRING STRETCHED ON A SURFACE.

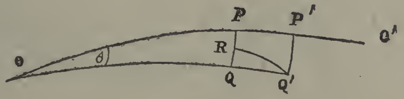
By *J. H. Michell, B.A.*, Scholar of Trinity College, Cambridge.

In his book on Sound Lord Rayleigh has considered the case of a string on a sphere.

With the use of Gauss's geodesic coordinates a simple solution of the question for any surface may be given.

Let the string lie along the geodesic OPO'.

The point P of the string during vibration is assumed to move along the geodesic circle through P and having O for its centre.



During the motion let the element PP' come to RQ' and draw the geodesic OQQ' meeting PR produced in Q.

We require the length RQ'.

Let $OP=x$, $PR=y$, and the angle $POR=\theta$. Let, further, P be Gauss's function, viz. $P\theta=y$; then P satisfies the equation

$$\frac{d^2P}{dx^2}+\frac{P}{RR'}=0,$$

where R, R' are the principal radii of curvature at P.

We have $$PP'=QQ'=dx,$$

$$PR=y,\ P'Q'=y+\frac{dy}{dx}dx,$$

$$\frac{PQ}{P'Q'}=\frac{P-P'dx}{P},\ \text{where } P'\equiv\frac{dP}{dx};$$

therefore $$PQ=\left(y+\frac{dy}{dx}dx\right)\left(1-\frac{P'}{P}dx\right);$$

therefore $$RQ=\left(\frac{dy}{dx}-y\frac{P'}{P}\right)dx;$$

therefore
$$RQ'^2 = (dx)^2 \left\{1 + \left(\frac{dy}{dx} - y\frac{P'}{P}\right)^2\right\},$$

so that the extension of dx is

$$\tfrac{1}{2}\left\{\frac{dy}{dx} - y\frac{P'(x)}{P(x)}\right\}^2 dx.$$

Hence the potential energy of the string is

$$V = \tfrac{1}{2}T\int\left[\frac{dy}{dx} - y\frac{P'}{P}\right]^2 dx,$$

where T is the tension.

Thererefore

$$\delta V = T\int\left[\frac{dy}{dx} - y\frac{P'}{P}\right]\left[\frac{d\delta y}{dx} - \delta y\frac{P'}{P}\right] dx$$

$$= T\left[\delta y\left(\frac{dy}{dx} - y\frac{P'}{P}\right)\right] - T\int\delta y\left[\frac{d}{dx}\left(\frac{dy}{dx} - y\frac{P'}{P}\right)\right.$$

$$\left. + \left(\frac{dy}{dx} - y\frac{P'}{P}\right)\frac{P'}{P}\right] dx.$$

The kinetic energy is $T_1 = \tfrac{1}{2}\rho\int\dot{y}^2\,dx$.

The equation of virtual velocities is

$$\rho\int\ddot{y}\,\delta y\,dx + \delta V = 0,$$

which gives the equation

$$\frac{\rho}{T}\ddot{y} - \frac{d^2y}{dx^2} + y\frac{P''(x)}{P} = 0,$$

or
$$\frac{\rho}{T}\ddot{y} = \frac{d^2y}{dx^2} + \frac{y}{RR'}.$$

Assuming a time factor e^{ipt}, we get

$$\frac{d^2y}{dx^2} + \frac{y}{RR'} + \frac{p^2\rho}{T}y = 0.$$

If RR' is constant along the geodesic, we have the solution

$$y = e^{i(pt+\kappa x)},$$

where
$$\kappa^2 = \frac{1}{RR'} + \frac{p^2\rho}{T}.$$

This agrees with Rayleigh's result for the sphere if we put $RR' = a^2$, a being the radius.

NOTES ON MODERN HIGHER ALGEBRA.

By *Thomas S. Fiske.*

I. *Converse of the Determinant Multiplication Theorem.*

Let a rational entire function, ψ, having n^2 arguments be subject to the same law of multiplication as a determinant, then is this function a determinant or a power of a determinant.

For, by hypothesis, we have

$$(1)\qquad \psi\left\{\begin{matrix} l_1, & l_2, & \ldots \\ m_1, & m_2, & \ldots \\ \multicolumn{3}{c}{\ldots\ldots\ldots\ldots} \end{matrix}\right\}.\psi\left\{\begin{matrix} \lambda_1, & \lambda_2, & \ldots \\ \mu_1, & \mu_2, & \ldots \\ \multicolumn{3}{c}{\ldots\ldots\ldots\ldots} \end{matrix}\right\}$$

$$=\psi\left\{\begin{matrix} l_1\lambda_1+l_2\lambda_2+\ldots, & l_1\mu_1+l_2\mu_2+\ldots, & \ldots \\ m_1\lambda_1+m_2\lambda_2+\ldots, & m_1\mu_1+m_2\mu_2+\ldots, & \ldots \\ \multicolumn{3}{c}{\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots} \end{matrix}\right\}.$$

A special case of this is

$$(2)\ \psi\left\{\begin{matrix} l_1, & l_2, & l_3, & \ldots \\ m_1, & m_2, & m_3, & \ldots \\ n_1, & n_2, & n_3, & \ldots \\ \multicolumn{4}{c}{\ldots\ldots\ldots\ldots} \end{matrix}\right\}.\psi\left\{\begin{matrix} \lambda_1, & 0, & 0, & \ldots \\ 0, & \mu_2, & 0, & \ldots \\ 0, & 0, & \nu_3, & \ldots \\ \multicolumn{4}{c}{\ldots\ldots\ldots\ldots} \end{matrix}\right\}=\psi\left\{\begin{matrix} l_1\lambda_1, & l_2\mu_2, & l_3\nu_3, & \ldots \\ m_1\lambda_1, & m_2\mu_2, & m_3\nu_3, & \ldots \\ n_1\lambda_1, & n_2\mu_2, & n_3\nu_3, & \ldots \\ \multicolumn{4}{c}{\ldots\ldots\ldots\ldots\ldots\ldots} \end{matrix}\right\},$$

from which it follows that ψ is homogeneous in the constituents of any one of its columns. The second member of (1) will vanish, therefore, if we make the supposition that the constituents of its first column are equal to zero, viz.

$$(3)\qquad \begin{aligned} l_1\lambda_1+l_2\lambda_2+l_3\lambda_3+\ldots&=0,\\ m_1\lambda_1+m_2\lambda_2+m_3\lambda_3+\ldots&=0,\\ n_1\lambda_1+n_2\lambda_2+n_3\lambda_3+\ldots&=0,\\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots& \end{aligned}$$

The further supposition that $\lambda_1, \lambda_2, \lambda_3, \ldots$ are not equal to zero, requires

$$(4)\qquad \left|\begin{matrix} l_1, & l_2, & l_3, & \ldots \\ m_1, & m_2, & m_3, & \ldots \\ n_1, & n_2, & n_3, & \ldots \\ \multicolumn{4}{c}{\ldots\ldots\ldots\ldots\ldots\ldots} \end{matrix}\right|=0.$$

But the result of the preceding condition is

$$(5)\qquad \psi\left\{\begin{matrix} l_1, & l_2, & \ldots \\ m_1, & m_2, & \ldots \\ \multicolumn{3}{c}{\ldots\ldots\ldots\ldots} \end{matrix}\right\}.\psi\left\{\begin{matrix} \lambda_1, & \lambda_2, & \ldots \\ \mu_1, & \mu_2, & \ldots \\ \multicolumn{3}{c}{\ldots\ldots\ldots\ldots} \end{matrix}\right\}=0.$$

The first factor of this product therefore contains the above determinant as a factor. Hence, the function ψ either is a determinant, or is equal to a determinant multiplied by a function which possesses the same fundamental property as itself. It is then a power of a determinant.

II. *The first proposition in the Theory of Concomitants.*

If we adopt the following definition—A concomitant is a rational entire function related to a quantic or a system of quantics so that when the variables connected therewith suffer linear transformation, it will be contained as a factor in the function related in the same way to the transformed quantic or quantics—the first proposition to be proved is: The transformed concomitant is equal to the original concomitant multiplied by a power of the modulus of transformation.

Demonstrations of this proposition have been given by M. Jordan in his *Calcul Différentiel* and by Mr. E. B. Elliot in vol. XVI. of this journal. The principle of the preceding section leads at once to a demonstration on different lines.

Let f denote a concomitant and let F denote its value after transformation. By definition, we have

$$F = \phi . f, \tag{6}$$

where the nature of ϕ is to be determined.

Since all the elements of F, coefficients and variables of the transformed quantic or quantics, are linear in the corresponding elements of f, and since F and f are similar functional symbols, the factor ϕ can involve only the constants of transformation. Further, no fractions can enter ϕ except those which result from the expressions for the new variables in terms of the old ones, and since these all have for a denominator the modulus of transformation, D, ϕ is reducible to a single fraction having a power of D for its denominator. Hence

$$F = \frac{\psi}{D^r} . f, \tag{7}$$

where ψ is an entire function of the constants of transformation.

The two successive transformations

$$\begin{aligned} x_1 &= l_1 X_1 + l_2 X_2 + \ldots, & X_1 &= \lambda_1 \xi_1 + \lambda_2 \xi_2 + \ldots, \\ x_2 &= m_1 X_1 + m_2 X_2 + \ldots, & X_2 &= \mu_1 \xi_1 + \mu_2 \xi_2 + \ldots, \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \end{aligned}$$

are equivalent to the single transformation

$$\begin{aligned} x_1 &= (l_1\lambda_1 + l_2\mu_1 + \ldots)\,\xi_1 + (l_1\lambda_2 + l_2\mu_2 + \ldots)\,\eta_2 + \ldots, \\ x_2 &= (m_1\lambda_1 + m_2\mu_1 + \ldots)\,\xi_1 + (m_1\lambda_2 + m_2\mu_2 + \ldots)\,\xi_2 + \ldots, \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned}$$

Taking these in connection with (7), we have

$$\text{(8)}\qquad \psi\left\{\begin{matrix} l_1\lambda_1 + l_2\mu_1 + \ldots, & l_1\lambda_2 + l_2\mu_2 + \ldots, & \ldots \\ m_1\lambda_1 + m_2\mu_1 + \ldots, & m_1\lambda_2 + m_2\mu_2 + \ldots, & \ldots \\ \ldots\ldots\ldots\ldots & \ldots\ldots\ldots\ldots & \ldots \end{matrix}\right\}$$

$$= \psi\left\{\begin{matrix} \lambda_1, & \lambda_2, & \ldots \\ \mu_1, & \mu_2, & \ldots \\ \ldots & \ldots & \ldots \end{matrix}\right\} . \psi\left\{\begin{matrix} l_1, & l_2, & \ldots \\ m_1, & m_2, & \ldots \\ \ldots & \ldots & \ldots \end{matrix}\right\} .$$

Here the multiplication of ψ is in accordance with the law of determinant multiplication. By the preceding section it follows that ψ is a power of the modulus of transformation. The same is true of ϕ.

GORDAN'S SERIES.

By *H. F. Baker*, St. John's College, Cambridge.

Introduction. As an example in Elementary Algebra it is possible that the following may be worthy a place in the *Messenger of Mathematics*. It is modified from Paul Gordan, "Ueber Combinanten," in the *Math. Annal.*, Bd. V., p. 95; and may form an elementary introduction in English to Gordan's Series.

§ 1. Let m, n be two positive integers; denote the less by μ and consider two sets of $\mu + 1$ quantities $A_0 \ldots A_i \ldots A_\mu$, $B_0 \ldots B_i \ldots B_\mu$, which are so connected that, when B_j is throughout replaced by $(m-j)(n-j)B_{j+1} + j(p-j+1)B_j$, A_i becomes replaced by $i(p-i+1)A_i$ (for all values of j and i), p being a positive integer. We proceed to shew that these relations can be uniquely expressed, independently of the actual values of the quantities A, B, by series of the form

$$(-1)^i A_i = \sum_{j=i}^{j=\mu} (-1)^j e_{i,j} B_j, \quad B_i = \sum_{j=i}^{j=\mu} c_{j,i} A_j$$

(where we take $e_{i,i} = c_{j,j} = 1$).

For, applying the relations of condition to the first series, we obtain

$$i(p-i+1)\sum_{j=i}^{j=\mu}(-1)^j e_{i,j}B_j=\sum_{j=i}^{j=\mu}(-1)^j e_{i,j}\,j(p-j+1)\,B_j$$
$$+\sum_{j=i}^{j=\mu}(-1)^j e_{i,j}(n-j)(m-j)\,B_{j+1},$$

and the last term of the second summation on the right vanishes. Put then here $j=j-1$, and this summation becomes

$$-\sum_{j=i+1}^{j=\mu}(-1)^j e_{i,j-1}(n-j+1)(m-j+1)\,B_j.$$

Equating herein the coefficients of B_j (for $j>i$, $j=i$ gives an identity) we have

$$i(p-i+1)\,e_{i,j}=e_{i,j}.j(p-j+1)-e_{i,j-1}(n-j+1)(m-j+1),$$

or $$e_{i,j}=\frac{(n-j+1)(m-j+1)}{(j-i)(p-j-i+1)}\,e_{i,j-1}$$
$$=\frac{(n-j+1)(n-j+2).(m-j+1)(m-j+2)}{(j-i)(j-i-1).(p-j-i+1)(p-j-i+2)}\,e_{i,j-2}.$$

Proceeding thus, by continued reduction to $e_{i,i}=1$, we have

$$e_{i,j}=\frac{\binom{n-i}{j-i}\binom{m-i}{j-i}}{\binom{p-2i}{j-i}},$$

where the notation $\binom{t}{s}$ is used for $\dfrac{t!}{s!\,(t-s)!}$ and p is assumed greater than $2\mu-2$, and therefore

$$(-1)^i A_i=\sum_{j=i}^{j=\mu}(-1)^j\frac{\binom{n-i}{j-i}\binom{m-i}{j-i}}{\binom{p-2i}{j-i}}B_j\,\ldots\ldots.(\mathrm{I}),$$

which is the first set of series.

From the last of these $\mu+1$ equations we have an expression for B_μ in terms of A_μ alone ($B_\mu=A_\mu$). From the last but one we have $B_{\mu-1}$ expressed by A_μ and $A_{\mu-1}$, and so on. There is then a unique expression of the form

$$B_i=\sum_{j=i}^{j=\mu}c_{j,i}A_j.$$

Applying the equations of condition to these, we obtain

$$c_{j,i} = \frac{(n-i)(m-i)}{(j-i)(p-j-i+1)} c_{j,i+1} \qquad (j > i)$$

$$= \frac{\binom{n-i}{j-i}\binom{m-i}{j-i}}{\binom{p-j-i+1}{j-i}} c_{j,j};$$

and therefore

$$B_i = \sum_{j=i}^{j=\mu} \frac{\binom{n-i}{j-i}\binom{m-i}{j-i}}{\binom{p-j-i+1}{j-i}} A_j \text{...............(II)},$$

and this is the second set of series, of course equivalent with the first.

If we substitute the values of A_i, obtained from (I), in (II), and equate the coefficients of B_k on both sides, we find the identity

$$\sum_{j=i}^{j=k} (-1)^j \frac{\binom{n-i}{j-i}\binom{m-i}{j-i}}{\binom{p-j-i+1}{j-i}} \frac{\binom{n-j}{k-j}\binom{m-j}{k-j}}{\binom{p-2j}{k-j}} = 0,$$

which can be reduced to

$$\sum_{j=0}^{j=k} (-1)^j \frac{\left[\binom{k}{j}\right]^2}{\binom{p-j+1}{j}\binom{p-2j}{k-j}} = 0,$$

where the notation is changed, or further, to

$$\sum_{j=0}^{j=k} (-1)^j (p-2j) \frac{\binom{k}{j}}{\binom{p-j}{k+1}} = 0 \text{ ....(III)},$$

which, for example, for $k=3$, leads to

$$(p-4)(p-5)-3(p-2)(p-5)+3(p-1)(p-4)-(p-1)(p-2)=0.$$

If we substitute the values of B_i, obtained from (II), in (I), and equate the coefficients of A_k on both sides, we obtain

$$\sum_{j=i}^{j=k} (-1)^j \frac{\binom{n-i}{j-i}\binom{m-i}{j-i}}{\binom{p-2i}{j-i}} \frac{\binom{n-j}{k-j}\binom{m-j}{k-j}}{\binom{p-k-j+1}{k-j}} = 0,$$

which can be reduced to

$$\sum_{j=0}^{j=k} (-1)^j \binom{p-k+1}{j}\binom{p-j}{k-j} = 0,$$

and expresses that the coefficient of x^k in $(1+x)^{p-k+1}(1+x)^{-(p-k+1)}$ is zero.

§ 2. Putting now $\binom{n+m}{r} C_r(x^n y^n)$ for the coefficient of λ^r in $(x+\lambda\xi)^n (y+\lambda\eta)^m$ and $D = x\eta - y\xi$, we have

$$C_r(x^n y^m) = \sum_{i+k=r} \frac{\binom{n}{i}\binom{m}{k}}{\binom{n+m}{r}} x^{n-i}\xi^i y^{m-k}\eta^k;$$

and therefore

$$C_m(x^n y^m) = \sum_{i+k=m} \frac{\binom{n}{i}\binom{m}{k}}{\binom{m+n}{m}} x^{n-i}\eta^{m-i}(x\eta - D)^i,$$

from which we see that the quantities $D^i C_{m-i}(x^{n-i}y^{m-i})$ are such linear functions of the quantities $D^i x^{n-i}\eta^{m-i}$ as are the quantities A_i of the quantities B_i in the equation (I). To see that the numerical coefficients are the same, we have only to notice that, while

$$D\left(\frac{\partial^2}{\partial x\,\partial\eta} - \frac{\partial^2}{\partial y\,\partial\xi}\right)\{D^i C_{m-i}(x^{n-i}y^{m-i})\}$$

$$= i(n+m-i+1)\,D^i C_{m-i}(x^{n-i}y^{m-i}),$$

also

$$D\left(\frac{\partial^2}{\partial x\,\partial\eta} - \frac{\partial^2}{\partial y\,\partial\xi}\right)\{x^{n-i}\eta^{m-i}D^i\}$$

$$= (n-i)(m-i)\,D^{i+1}x^{n-i-1}\eta^{m-i-1} + i(n+m-i+1)\,D^i x^{n-i}\eta^{m-i},$$

which are just the equations of condition for A_i, B_i (putting therein $p=n+m$). Thus we have immediately

$$(-1)^i D^i C_{m-i}(x^{n-i}y^{m-i}) = \sum_{j=i}^{j=\mu} (-1)^j \frac{\binom{n-i}{j-i}\binom{m-i}{j-i}}{\binom{p-2i}{j-i}} D^j x^{n-j}\eta^{m-j} \ldots \text{(IV)},$$

and $$D^i x^{n-i}\eta^{m-i} = \sum_{j=i}^{j=\mu} \frac{\binom{n-i}{j-i}\binom{m-i}{j-i}}{\binom{p-j-i+1}{j-i}} D^j C_{m-j}(x^{n-j}y^{m-j}) \ldots \text{(V)},$$

$$\text{(for } p=n+m\text{)},$$

of which it is quite sufficient to take the two series arising for $i=0$ on the left hand; the others are deducible by change of n and m into $n-i$ and $m-i$.

In particular, putting $x=a_x=a_1x_1+a_2x_2$, $\xi=a_y$, $y=b_x$, $\eta=b_y$, so that D becomes

$$(ab)(xy)=(a_1b_2-a_2b_1)(x_1y_2-x_2y_1),$$

and $\binom{n+m}{r} C_r(x^n y^m)$ becomes coefficient of λ^r in

$$(a_x+\lambda a_y)^n (b_x+\lambda b_y)^m,$$

namely, coefficient of λ^r in the result of writing $x_i+\lambda y_i$ for x_i in $a_x{}^n b_x{}^m$, namely, $\frac{\Delta^r}{r!}(a_x{}^n b_x{}^m)$, where

$$\Delta = y_1\frac{\partial}{\partial x_1} + y_2\frac{\partial}{\partial x_2},$$

so that $(ab)^i C_{m-i}(x^{n-i}y^{m-i})$ becomes

$$(ab)^i \frac{(n-i)!}{(m+n-2i)!} \Delta^{m-i}(a_x{}^{n-i} b_x{}^{m-i}),$$

which we write $(a, b)^i_{y^{m-i}}$; and, putting $i=0$ on the left-hand of series (IV) and (V), we have

$$(a, b)^0_{y^m} = \sum_{j=0}^{j=\mu} (-1)^j \frac{\binom{n}{j}\binom{m}{j}}{\binom{p+m}{j}} (xy)^j (ab)^j a_x{}^{n-j} b_y{}^{m-j} \ldots \text{(VI)},$$

and $$a_x^n b_y^m = \sum_{j=0}^{j=\mu} \frac{\binom{n}{j}\binom{m}{j}}{\binom{m+n-j+1}{j}} (xy)^j (a, b)^j_{y^{m-j}} \dots \text{(VII)},$$

of which (VII) is the important formula which enables us to express forms such as $a_x^n b_y^m$ in two variables by differentials such as $(a, b)^j_{y^{m-j}}$ of forms of one variable.

§ 3. Further, we immediately see, referring to Gordan's article* for the notation, that

$$u_x \delta \{u_x^i [\Theta_i]\} = i (m+n+p-2-i+1) u_x^i [\Theta_i],$$

and

$$u_x \delta \{u_x^i \ \Theta_i\} = (n-i)(m-i) u_x^{i+1} \Theta_{i+1} + i (p+m+n-i-1) u_x^i \Theta_i,$$

so that the functions $u_x^i [\Theta_i]$, $u_x^i \Theta_i$ obey the same equations of condition as our quantities A_i and B_i respectively (by replacing our p by $m+n+p-2$). Thus the series (I) and (II) are immediately applicable to them.

Note. The equation (III) expresses that if

$$f(x) = (x-a_0)(x-a_1) \dots (x-a_k) \text{ and } a_i = i(p-i),$$

then $\Sigma \dfrac{1}{f'(a)} = 0$, the summation extending over the roots, a, of $f(x) = 0$.

MATHEMATICAL NOTES.

By *W. Burnside, M.A.*

1. *Geometrical interpretation of a condition of integrability.* The relation

$$l\left(\frac{\partial n}{\partial y} - \frac{\partial m}{\partial z}\right) + m\left(\frac{\partial l}{\partial z} - \frac{\partial n}{\partial x}\right) + n\left(\frac{\partial m}{\partial x} - \frac{\partial l}{\partial y}\right) = 0 \dots (A)$$

depends only on the ratios of l, m, n, and not on their absolute magnitudes.

Suppose then $l^2 + m^2 + n^2 = 1$, and choose any other pair of sets

$$l', m', n' \text{ and } l'', m'', n''$$

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* *Math. Annal.*, Bd. v., p. 102.

to form with l, m, n a mutually rectangular system. Then, these being taken in a proper cyclical order,

$$l=m'n''-m''n', \quad m=n'l''-n''l', \quad n=l'm''-l''m';$$

and hence

$$\begin{aligned} m\frac{\partial}{\partial z}-n\frac{\partial}{\partial y} &= (n'l''-n''l')\frac{\partial}{\partial z}-(l'm''-l''m')\frac{\partial}{\partial y} \\ &= l''\left(l'\frac{\partial}{\partial x}+m'\frac{\partial}{\partial y}+n'\frac{\partial}{\partial z}\right)-l'\left(l''\frac{\partial}{\partial x}+m''\frac{\partial}{\partial y}+n''\frac{\partial}{\partial z}\right) \\ &= l''\frac{\partial}{\partial h'}-l'\frac{\partial}{\partial h''}, \end{aligned}$$

where $\dfrac{\partial}{\partial h}$ is the operation of differentiating in the direction l, m, n.

Hence equation (A) may be written

$$l''\frac{\partial l}{\partial h'}+m''\frac{\partial m}{\partial h'}+n''\frac{\partial n}{\partial h'}=l'\frac{\partial l}{\partial h''}+m'\frac{\partial m}{\partial h''}+n'\frac{\partial n}{\partial h''}\ldots(B),$$

and in this form a simple geometrical meaning at once suggests itself.

Thus, if OA is the line through O, whose direction-cosines are l, m, n; OB, OC being $l'm'n'$, $l''m''n''$; and if equal small lengths OO', OO'' be taken on the latter lines, and lines $O'A'$, $O''A''$ of the system l, m, n be drawn through O', O''; then

$$OB\overset{\wedge}{,}O'A'=OC\overset{\wedge}{,}O''A'',$$

the quantities on either side of equation (B) being the cosines of these angles divided by OO'; and the lines $O'A'$, $O''A''$ must be either both inside or both outside the solid angle formed by OA, OB, OC. Hence, if the plane AOB turns round OA in some position between the original positions of AOB and AOC, the line $O'A'$ will lie in the AOB, and then for the same position $O''A''$ lies in AOC.

Hence, if a system of curves is capable of being cut orthogonally by a system of surfaces, then at every point there must be two planes at right angles intersecting in the tangent line to the curve passing through the point which contain tangent lines to consecutive curves of the system. This condition, though necessary, is clearly not sufficient, as it is only a verbal statement of the relation

$$OB\overset{\wedge}{,}O'A'=OC\overset{\wedge}{,}O''A''$$

for a special position of the planes AOB, AOC.

2. *Propagation of energy in the electro-magnetic field.*

Professor Poynting's demonstration of the way in which energy is propagated in the electro-magnetic field takes an extremely simple form when written in quaternion notation; and with that notation the supposition that the medium is not isotropic introduces no complication. Writing E and H for the electric and magnetic forces, and eliminating from the equations which Maxwell gives in Vol. II., Chap. IX., everything except E and H, there results, on the supposition that no part of the field is in bodily motion,

$$K(\dot{E}) = V.\nabla H - 4\pi C(E),$$

$$\mu(\dot{H}) = -V.\nabla E,$$

where the terms following K, C, μ are written in brackets to indicate that these are now symbols for linear and vector functions.

The energy per unit volume is given by

$$W = -\frac{1}{8\pi} S[EK(E) + H\mu(H)];$$

therefore $$\dot{W} = -\frac{1}{4\pi} S[EK(\dot{E}) + H\mu(\dot{H})]$$

since the functions K and μ are always self-conjugate; hence

$$\dot{W} = -\frac{1}{4\pi} S[E\{V.\nabla H - 4\pi C(E)\} - HV.\nabla E],$$

or $$\dot{W} - S.EC(E) = -\frac{1}{4\pi} S[E\nabla H - H\nabla E]$$

$$= -\frac{1}{4\pi} S.\nabla V.HE,$$

and from this form the result may be put into words, or perhaps more readily from

$$\iiint\{\dot{W} - S.EC(E)\}\,dv = \iint S\frac{V.EH}{4\pi}U\nu\,ds.$$

THE LINES OF ZERO LENGTH ON A SURFACE AS CURVILINEAR COORDINATES.

By *W. Burnside, M.A.*

The following collection of formulæ and equations connected with a surface expressed in terms of the parameters of lines of zero length on the surface as independent variables contains, I believe, no new results; but may be useful as presenting concisely what otherwise has to be sought in many different places.

The position of any point x, y, z on a surface is supposed to be expressed in terms of two parameters η, η'; and the notation defined by the following equations is used:

$$\frac{\partial y}{\partial \eta}\frac{\partial z}{\partial \eta'} - \frac{\partial y}{\partial \eta'}\frac{\partial z}{\partial \eta} = \lambda,\ \frac{\partial z}{\partial \eta}\frac{\partial x}{\partial \eta'} - \frac{\partial z}{\partial \eta'}\frac{\partial x}{\partial \eta} = \mu,\ \frac{\partial x}{\partial \eta}\frac{\partial y}{\partial \eta'} - \frac{\partial x}{\partial \eta'}\frac{\partial y}{\partial \eta} = \nu,$$

$$\left(\frac{\partial x}{\partial \eta}\right)^2 + \left(\frac{\partial y}{\partial \eta}\right)^2 + \left(\frac{\partial z}{\partial \eta}\right)^2 = s,\quad \left(\frac{\partial x}{\partial \eta'}\right)^2 + \left(\frac{\partial y}{\partial \eta'}\right)^2 + \left(\frac{\partial z}{\partial \eta'}\right)^2 = s',$$

$$\frac{\partial x}{\partial \eta}\frac{\partial x}{\partial \eta'} + \frac{\partial y}{\partial \eta}\frac{\partial y}{\partial \eta'} + \frac{\partial z}{\partial \eta}\frac{\partial z}{\partial \eta'} = s'',$$

$$\lambda\frac{\partial^2 x}{\partial \eta^2} + \mu\frac{\partial^2 y}{\partial \eta^2} + \nu\frac{\partial^2 z}{\partial \eta^2} = p,\quad \lambda\frac{\partial^2 x}{\partial \eta'^2} + \mu\frac{\partial^2 y}{\partial \eta'^2} + \nu\frac{\partial^2 z}{\partial \eta'^2} = p',$$

$$\lambda\frac{\partial^2 x}{\partial \eta\,\partial \eta'} + \mu\frac{\partial^2 y}{\partial \eta\,\partial \eta'} + \nu\frac{\partial^2 z}{\partial \eta\,\partial \eta'} = p''.$$

The only algebraical relation between the nine quantities so defined is

$$\lambda^2 + \mu^2 + \nu^2 = ss' - s''^2.$$

The squared element of length on the surface is

$$s\,d\eta^2 + s'\,d\eta'^2 + 2s''\,d\eta\,d\eta'.$$

The equation of the lines of curvature

$$\begin{vmatrix} dx, & dy, & dz \\ d\lambda, & d\mu, & d\nu \\ \lambda, & \mu, & \nu \end{vmatrix} = 0,$$

expressed in terms of these quantities becomes

$$(sp'' - s''p)\,d\eta^2 + (sp' - s'p)\,d\eta\,d\eta' + (s''p' - s'p'')\,d\eta'^2 = 0.$$

The asymptotic lines (lines whose osculating plane coincides with the tangent plane to the surface)

$$\begin{vmatrix} d^2x, & d^2y, & d^2z \\ \frac{\partial x}{\partial \eta}, & \frac{\partial y}{\partial \eta}, & \frac{\partial z}{\partial \eta} \\ \frac{\partial x}{\partial \eta'}, & \frac{\partial y}{\partial \eta'}, & \frac{\partial z}{\partial \eta'} \end{vmatrix} = 0$$

take the form

$$p\,d\eta^2 + 2p''d\eta\,d\eta' + p'\,d\eta'^2 = 0.$$

The radius of curvature of a normal section is given by

$$\frac{\sqrt{(ss' - s''^2)}}{\rho} = \frac{p\,d\eta^2 + 2p''d\eta\,d\eta' + p'd\eta'^2}{s\,d\eta^2 + 2s''\,d\eta\,d\eta' + s'\,d\eta'^2},$$

whence the sum and product of the principal curvatures satisfy

$$\frac{1}{R} + \frac{1}{R'} = \frac{ps' + p's - 2p''s''}{(ss' - s''^2)^{\frac{3}{2}}},$$

$$\frac{1}{RR'} = \frac{pp' - p''^2}{(ss' - s''^2)^2}.$$

The proofs of the above forms are omitted as being either familiar, or obtainable by simple algebraical processes.

If $\eta = \text{const.}$, $\eta' = \text{const.}$ are the two families of lines of zero length on the surface, then $s = s' = 0$.

When the surface is real, η and η' may always be chosen so as to be conjugate imaginaries.

When this is the case s'' is real, and λ, μ, ν are therefore pure imaginaries.

Also $\frac{\partial^2 x}{\partial \eta^2}$ and $\frac{\partial^2 x}{\partial \eta'^2}$ will be conjugate imaginaries and $\frac{\partial^2 x}{\partial \eta\,\partial \eta'}$ will be real, so that p and p' are such that their imaginary parts are equal and their real parts equal and opposite, while p'' is a pure imaginary. The equation $s = 0$ gives, on differentiating with respect to η,

$$\left(\frac{\partial x}{\partial \eta}\frac{\partial^2 x}{\partial \eta^2}\right)^2 = \left(\frac{\partial y}{\partial \eta}\frac{\partial^2 y}{\partial \eta^2} + \frac{\partial z}{\partial \eta}\frac{\partial^2 z}{\partial \eta^2}\right)^2$$

$$= \left\{\left(\frac{\partial y}{\partial \eta}\right)^2 + \left(\frac{\partial z}{\partial \eta}\right)^2\right\}\left\{\left(\frac{\partial^2 y}{\partial \eta^2}\right)^2 + \left(\frac{\partial^2 z}{\partial \eta^2}\right)^2\right\} - \left(\frac{\partial y}{\partial \eta}\frac{\partial^2 z}{\partial \eta^2} - \frac{\partial z}{\partial \eta}\frac{\partial^2 y}{\partial \eta^2}\right)^2;$$

therefore

$$\left(\frac{\partial y}{\partial \eta}\frac{\partial^2 z}{\partial \eta^2}-\frac{\partial z}{\partial \eta}\frac{\partial^2 y}{\partial \eta^2}\right)^2=\left(\frac{\partial x}{\partial \eta}\right)^2\left\{\left(\frac{\partial^2 x}{\partial \eta^2}\right)^2+\left(\frac{\partial^2 y}{\partial \eta^2}\right)^2+\left(\frac{\partial^2 z}{\partial \eta^2}\right)^2\right\},$$

and $$\frac{\partial y}{\partial \eta}\frac{\partial^2 z}{\partial \eta^2}-\frac{\partial z}{\partial \eta}\frac{\partial^2 y}{\partial \eta^2}=-i\frac{\partial x}{\partial \eta}q, \text{ say;}$$

therefore $$p=is''q,$$

and similarly $$p'=is''q',$$

$$p''=is''q'',$$

where q, q', q'' are those square roots of

$$\left(\frac{\partial^2 x}{\partial \eta^2}\right)^2+\left(\frac{\partial^2 y}{\partial \eta^2}\right)^2+\left(\frac{\partial^2 z}{\partial \eta^2}\right)^2, \quad \left(\frac{\partial^2 x}{\partial \eta'^2}\right)^2+\left(\frac{\partial^2 y}{\partial \eta'^2}\right)^2+\left(\frac{\partial^2 z}{\partial \eta'^2}\right)^2,$$

and $$\left(\frac{\partial^2 x}{\partial \eta\,\partial \eta'}\right)^2+\left(\frac{\partial^2 y}{\partial \eta\,\partial \eta'}\right)^2+\left(\frac{\partial^2 z}{\partial \eta\,\partial \eta'}\right)^2,$$

which have their real parts positive.

It will be found that this determination of the signs agrees with the statements made above with respect to p, p', and p''.

The equations of the lines of curvature and asymptotic lines become now

$$q\,d\eta^2-q'\,d\eta'^2=0,$$

and $$q\,d\eta^2+q'\,d\eta'^2+2q''\,d\eta\,d\eta''=0,$$

respectively.

The sum of the principal curvatures is given by

$$\frac{1}{R}+\frac{1}{R'}=2\frac{q''}{s''},$$

and the measure of curvature by

$$\frac{1}{RR'}=\frac{q''^2-qq'}{s''^2}.$$

It is not difficult to verify that the measure of curvature is, as it should be, expressible in terms of s''.

Thus $$\frac{\partial s''}{\partial \eta}=\frac{\partial^2 x}{\partial \eta^2}\frac{\partial x}{\partial \eta'}+\ldots+\frac{\partial x}{\partial \eta}\frac{\partial^2 x}{\partial \eta\,\partial \eta'}+\ldots$$

$$=\frac{\partial^2 x}{\partial \eta^2}\frac{\partial x}{\partial \eta'}+\ldots, \text{ since } \frac{\partial s}{\partial \eta'}=0,$$

so
$$\frac{\partial s''}{\partial \eta'} = \frac{\partial^2 x}{\partial \eta'^2}\,\frac{\partial x}{\partial \eta} + \dots,$$

and
$$\frac{\partial^2 s''}{\partial\eta\,\partial\eta'} = \frac{\partial^2 x}{\partial\eta^2}\,\frac{\partial^2 x}{\partial\eta'^2} + \dots + \frac{\partial^3 x}{\partial\eta^2\,\partial\eta'}\,\frac{\partial x}{\partial\eta'} + \dots$$
$$= \frac{\partial^2 x}{\partial\eta^2}\,\frac{\partial^2 x}{\partial\eta'^2} + \dots - \left(\frac{\partial^2 x}{\partial\eta\,\partial\eta'}\right)^2 - \dots,$$

since
$$\frac{\partial^2 s'}{\partial\eta^2} = 0\,;$$

therefore
$$\frac{\partial s''}{\partial\eta}\,\frac{\partial s''}{\partial\eta'} - s''\,\frac{\partial^2 s''}{\partial\eta\,\partial\eta'} - s''q''^2 = \left(\frac{\partial^2 x}{\partial\eta^2}\,\frac{\partial x}{\partial\eta'} + \dots\right)\left(\frac{\partial^2 x}{\partial\eta'^2}\,\frac{\partial x}{\partial\eta} + \dots\right)$$
$$- \left(\frac{\partial x}{\partial\eta}\,\frac{\partial x}{\partial\eta'} + \dots\right)\left(\frac{\partial^2 x}{\partial\eta^2}\,\frac{\partial^2 x}{\partial\eta'^2} + \dots\right)$$
$$= -\left(\frac{\partial y}{\partial\eta}\,\frac{\partial^2 z}{\partial\eta^2} - \frac{\partial z}{\partial\eta}\,\frac{\partial^2 y}{\partial\eta^2}\right)\left(\frac{\partial y}{\partial\eta'}\,\frac{\partial^2 z}{\partial\eta'^2} - \frac{\partial z}{\partial\eta'}\,\frac{\partial^2 y}{\partial\eta'^2}\right) - \dots$$
$$= -\left(-i\,\frac{\partial x}{\partial\eta}\,q\right)\left(i\,\frac{\partial x}{\partial\eta'}\,q'\right) - \dots$$
$$= -\,s''qq'\,;$$

therefore
$$\frac{1}{RR'} = \frac{\dfrac{\partial s''}{\partial\eta}\,\dfrac{\partial s''}{\partial\eta'} - s''\,\dfrac{\partial^2 s''}{\partial\eta\,\partial\eta'}}{s''^3} = -\frac{1}{s''}\,\frac{\partial^2}{\partial\eta\,\partial\eta'}\log s''.$$

The length of an arc on the surface being
$$\int\sqrt{(2s''d\eta\,d\eta')},$$
the differential equation to the geodesics will be obtained from
$$\delta\int\sqrt{(2s''d\eta\,d\eta')} = 0,$$
which gives
$$s''\,\frac{d^2\eta'}{d\eta^2} - \left(\frac{\partial s''}{\partial\eta} - \frac{\partial s''}{\partial\eta'}\,\frac{d\eta'}{d\eta}\right)\frac{d\eta'}{d\eta} = 0.$$

In this form it is evident that the lines of zero length are themselves geodesics.

It is to minimum-surfaces, that is, surfaces such that their area is a minimum subject to certain boundary conditions, that the above forms and equations are chiefly applied.

These surfaces are such that the sum of the principal curvatures is everywhere zero. It is interesting, however, to express directly that the variation of the area vanishes. Thus

$$\delta \iint \sqrt{(\lambda^2 + \mu^2 + \nu^2)}\, d\eta\, d\eta' = 0$$

leads at once to

$$\frac{\partial}{\partial \eta} \frac{\mu \dfrac{\partial z}{\partial \eta'} - \nu \dfrac{\partial \eta}{\partial \eta'}}{\sqrt{(\lambda^2 + \mu^2 + \nu^2)}} - \frac{\partial}{\partial \eta'} \frac{\mu \dfrac{\partial z}{\partial \eta} - \nu \dfrac{\partial y}{\partial \eta}}{\sqrt{(\lambda^2 + \mu^2 + \nu^2)}} = 0,$$

and two similar equations.

These may be written

$$\frac{\partial}{\partial \eta} \frac{s' \dfrac{\partial x}{\partial \eta} - s'' \dfrac{\partial x}{\partial \eta'}}{\sqrt{(ss' - s''^2)}} + \frac{\partial}{\partial \eta'} \frac{s \dfrac{\partial x}{\partial \eta'} - s'' \dfrac{\partial x}{\partial \eta}}{\sqrt{(ss' - s''^2)}} = 0,$$

$$\dots\dots\dots\dots\dots\dots,$$

$$\dots\dots\dots\dots\dots\dots,$$

and these, when expanded, give

$$\lambda\, (s'p + sp - 2s''p'') = 0, \ \dots, \ \dots,$$

equivalent to the single equation

$$\frac{1}{R} + \frac{1}{R'} = 0.$$

But if now again, η and η' are the parameters of the lines of zero length, and $s = s' = 0$, the three equations expressing that the variation of the surface vanishes reduce directly to

$$\frac{\partial^2 x}{\partial \eta \partial \eta'} = 0, \quad \frac{\partial^2 y}{\partial \eta \partial \eta'} = 0, \quad \frac{\partial^2 z}{\partial \eta \partial \eta'} = 0.$$

The surface will therefore be given by

$$x = x_1 + x_2, \quad y = y_1 + y_2, \quad z = z_1 + z_2,$$

where x_1, y_1, z_1 are functions of η only, and x_2, y_2, z_2 functions of η' only, satisfying the equations

$$\left(\frac{\partial x_1}{\partial \eta}\right)^2 + \left(\frac{\partial y_1}{\partial \eta}\right)^2 + \left(\frac{\partial z_1}{\partial \eta}\right)^2 = 0,$$

$$\left(\frac{\partial x_2}{\partial \eta'}\right)^2 + \left(\frac{\partial y_2}{\partial \eta'}\right)^2 + \left(\frac{\partial z_2}{\partial \eta'}\right)^2 = 0.$$

It is easy to see that the most general values satisfying the first of these may be written

$$\frac{\partial x_1}{\partial \eta} = (1-\eta^2)f, \quad \frac{\partial y}{\partial \eta} = i(1+\eta^2)f, \quad \frac{\partial z}{\partial \eta} = 2\eta f,$$

where f is any function of η; and that therefore

$$\begin{aligned} x &= \int(1-\eta^2)f\,d\eta + \int(1-\eta'^2)\,F\,d\eta', \\ y &= i\int(1+\eta^2)f\,d\eta + i\int(1+\eta'^2)\,F\,d\eta', \\ z &= 2\int\eta f\,d\eta \qquad + 2\int\eta' F\,d\eta', \end{aligned}$$

are the most general values of x, y, z, satisfying the required conditions. These are the well-known forms of Weierstrass. If the surface is real the functions f and F must be conjugate imaginaries.

With this notation, q and q' become $2f$ and $2F$, q'' being zero.

The equations to the lines of curvature and the asymptotic lines are now integrable by quadratures, q being a function of η only and q' of η'.

Also

$$\begin{aligned} s'' &= \{(1-\eta^2)(1-\eta'^2) - (1+\eta^2)(1+\eta'^2) + 4\eta\eta'\}fF \\ &= -2(\eta-\eta')^2\,\tfrac{1}{4}(qq'), \end{aligned}$$

and therefore $$\frac{1}{RR'} = \frac{-4}{(\eta-\eta')^4\,qq'}.$$

Finally, the form of a real minimum surface is completely determinate when the function

$$\left(\frac{\partial^2 x}{\partial \eta^2}\right)^2 + \left(\frac{\partial^2 y}{\partial \eta^2}\right)^2 + \left(\frac{\partial^2 z}{\partial \eta^2}\right)^2$$

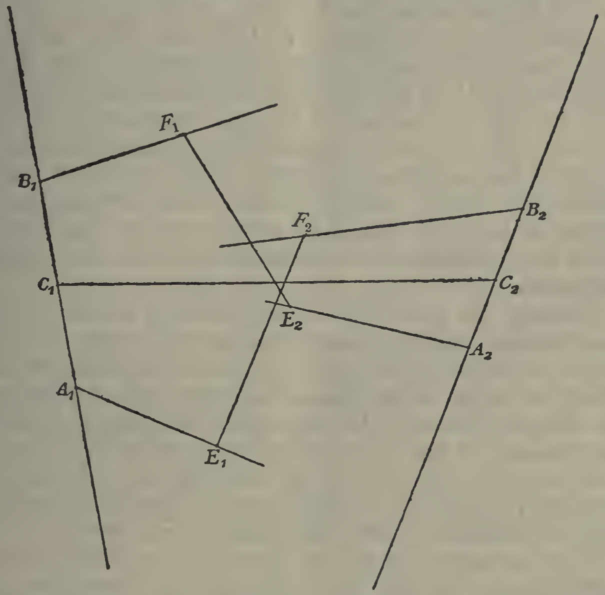
is given.

ON THE RESULTANT OF TWO FINITE DISPLACEMENTS OF A RIGID BODY.

By *W. Burnside, M.A.*

THE object of the following note is to state a simple geometrical construction for the resultant of any two finite displacements of a rigid body. The two given displacements may be specified as twists about certain screws, and what is sought will be the axis, translation and angle of the resultant twist.

Let A_1B_1, A_2B_2 be the axes of the given screws, the suffixes denoting the order in which the twists are made.



Draw the line C_1C_2 meeting A_1B_1, A_2B_2 at right angles, and lines A_1E_1, B_2F_2 such that half the given twist about A_1B_1 brings A_1E_1 to C_1C_2, and half the given twist about A_2B_2 brings C_2C_1 to B_2F_2. Finally draw the line E_1F_2 meeting A_1E_1, B_2F_2 at right angles. Then E_1F_2 is the axis of the resultant twist, $2E_1F_2$ is its translation, and twice the angle between A_1E_1 and F_2B_2 is its angle.

Complete the figure by drawing B_1F_1, A_2E_2, such that half the given twists will bring C_1C_2 to B_1F_1 and A_2E_2 to C_2C_1, and let E_2F_1 be the line at right angles to A_2E_2 and B_1F_1.

The figure thus obtained and that resulting from it by rotating the whole through two right angles about C_1C_2 coincide, whence it follows, if proof is necessary, that

$$A_1E_1 = B_1F_1,\ A_2E_2 = B_2F_2,\ E_1F_2 = E_2F_1,$$

and that the angles between the planes $C_1A_1E_1$, $A_1E_1F_2$, and $C_2A_2E_2$, $A_2E_2F_1$ are equal to those between the planes $C_1B_1F_1$, $B_1F_1E_2$, and $C_2B_2F_2$, $B_2F_2E_1$ respectively.

Hence, at once, the twist about A_1B_1 brings E_1F_2 to E_2F_1 produced, the point E_1 being brought to F_1; and then the twist about A_2B_2 brings E_2F_1 to E_1F_2 produced, E_2 being brought to F_2, so that E_1F_2 is the axis of the resultant twist and $2E_1F_2$ the translation.

Again, the angle between A_1E_1 and F_2B_2 being equal (from the same considerations as above) to that between B_1F_1 and E_2A_2; A_1E_1 which by the first twist is brought to B_1F_1 will by the second be brought into a position making the same angle with F_2B_2 that F_2B_2 makes with A_1E_1, and therefore the rotation is through twice the angle that F_2B_2 makes with A_1E_1. Clearly, if the twists were applied in the reverse order E_2F_1 would be the axis of the resultant twist, while the amount of translation and rotation would be unaltered.

Among the particular cases included in the foregoing construction may be noticed that in which one of the twists is a pure rotation and the other a translation. The construction then gives a simple method of reducing the displacement to its canonical form. It may also be remarked that the directions of all the lines in the figure are independent of the lengths A_1B_1, A_2B_2, C_1C_2, so that the general construction contains a proof of Rodrigue's theorem for the composition of finite rotation about intersecting axes.

The statement of the construction has been so worded as to bring out the fact that it is independent of the parallel-axiom, and that it will therefore apply to what is generally called non-Euclidean space. It thus enables the kinematics of hyperbolic and elliptic space, which have generally been reasoned above by purely analytical methods, to be treated geometrically.

When a body in non-Euclidean space receives a translation, it is only those points of the body which describe the axis of the translation that move in a straight line, all other points describing plane curves, called *equidistants* of the axis. It is not then to be expected that the resultant of two translations is necessarily a translation, and the construction shews that this is only the case when the axes of the translations are in a plane; and even in this case the usual triangle construction can only be applied when the axes intersect and with due regard to the order of the translations. In single elliptic space the construction is no longer absolutely determinate, since two different lines can be drawn at right angles to each of two given lines.

As the process of drawing a line at right angles to each of two given lines is gone through twice in the construction, it would seem as though here there would follow the inconsistent result of four possible axes for the resultant twist; but it is easily shewn as follows that only two of the four lines are really different, and that these are polars one of the other.

For suppose in the figure that $C_1'C_2'$, the other line meeting A_1B_1, A_2B_2 at right angles, is drawn (this is the polar line of C_1C_2), and complete the figure about the line $C_1'C_2'$, using accented letters.

Half the given twist about A_1B_1 will bring A_1E_1 and $A_1'E_1'$ to C_1C_2 and $C_1'C_2'$ respectively. Hence A_1E_1, $A_1'E_1'$ are polars, and so also are B_2F_2, $B_2'F_2'$. It follows that E_1F_2, meeting A_1E_1, B_2F_2 at right angles, also meets $A_1'E_1'$, $B_2'F_2'$ at right angles, and that the same holds for the polar line of E_1F_2, so that the construction leads only to E_1F_2 or its polar as the axis of the resultant twist.

If the displacements consist of equal translations along a line and its polar, then C_1 and C_2 are *any* points on A_1B_1 and A_2B_2; A_1E_1, B_1F_1 pass through C_2, and A_2E_2, B_2F_2 pass through C_1; also A_1B_1 being equal to A_2B_2, A_1E_1, B_2F_2 are straight lines such that their distance apart is everywhere the same, and an infinite number of lines can therefore be drawn at right angles to both of these. In this case the axis of the resultant twist is indeterminate, and since any one of the system of common perpendiculars to A_1E_1 and B_2F_2 may be chosen as the axis of the twist, the different points of the body move along this system of lines. Clifford (*Proc. Lon. Math. Soc.* IV., 389) has called this motion a vector, and the system of common perpendiculars to two lines such as A_1E_1, B_2F_2 parallels. Since when A_1B_1 is given A_2B_2 may be taken in either one direction or the other on the polar, there will be connected with a given line two systems of parallels, as Clifford has shewn.

If a body moves with equal translation velocities along two polar lines A_1B_1, A_2B_2, then any point P of the body describes a parallel to A_1B_1 and A_2B_2, and not until the points moving in A_1B_1, A_2B_2 have described the whole of those lines will P come back to its original position. If the displacements along A_1B_1, A_2B_2 be made in any way, P will not necessarily describe the parallel, but it still remains true that it will not return to its original position till translations through the whole lines A_1B_1, A_2B_2 are completed. It follows that a translation along a complete line A_1B_1 does not bring a body back to its original position; but since it brings all points in A_1B_1 back to their original positions it must be equivalent to a rotation about A_1B_1. By the simple properties of polar lines a translation along A_2B_2 is equivalent to a rotation round A_1B_1; and hence, since translations along the complete lines A_1B_1,

A_2B_2 are together equivalent to no displacement at all, the translation along the complete line A_1B_1 is equivalent to a rotation through two right angles round either it or A_2B_2. It follows that if the unit of length is such that the length of a complete line is π, then a translation through any distance along a line is equivalent to a rotation through an *equal angle* round the polar line. The best and most accessible short elementary exposition of the properties of non-Euclidean space may be found in a paper by Prof. Chrystal (*Proc. R. S. E.*, 1879–80), from which the nomenclature used and results assumed in this note are taken.

NOTE ON THE INTERSECTION OF LINES WITH CURVES OF AN ODD DEGREE.

By *H. G. Dawson, B.A.*, Fellow of Christ's College.

We start by proving the known result that if the line $y=mx+n$ cut $x^3+y^3=1$ in the points (x_1y_1), (x_2y_2), (x_3y_3), then $x_1x_2x_3+y_1y_2y_3=1$. The simplest way to establish the result seems to be as follows: substitute for y in the equation of the curve, and we get

$$x^3+(mx+n)^3=1,$$

therefore $$x_1x_2x_3=\frac{1-n^3}{1+m^3},$$

and $y_1y_2y_3$ will be got from this by changing m into $\frac{1}{m}$, and n into $-\frac{n}{m}$,

therefore $$y_1y_2y_3=\frac{m^3+n^3}{m^3+1},$$

therefore $$x_1x_2x_3+y_1y_2y_3=\frac{1+m^3}{1+m^3}=1.$$

A precisely similar theorem holds for the cubic

$$x^3+y^3+z^3+6mxyz=0;$$

viz., if $(x_1y_1z_1)$, $(x_2y_2z_2)$, $(x_3y_3z_3)$ are three collinear points on a cubic, then

$$x_1x_2x_3+y_1y_2y_3+z_1z_2z_3=0.$$

Solve for the ratio x/z between the equation of the cubic

$$x^3+y^3+z^3+6mxyz=0,$$

and an arbitrary line

$$\lambda x+\mu y+\nu z=0.$$

The equation for the ratio in question is

$$\mu^3(x^3+z^3)-(\lambda x+\nu z)^3-6m\mu^2 xz(\lambda x+\nu z)=0,$$

therefore $x_1x_2x_3/z_1z_2z_3=(\mu^3-\nu^3)/(\lambda^3-\mu^3)$.

Similarly $y_1y_2y_3/z_1z_2z_3=(\lambda^3-\nu^3)/(\mu^3-\lambda^3)$,

and therefore $x_1x_2x_3+y_1y_2y_3+z_1z_2z_3=0$,

a remarkable equation for it is independent of m,

Now it is well known that if P be a point on a cubic every chord through P is cut harmonically by the curve and by the polar conic of P. Hence if P, Q, R be a collinear triad of points on a cubic, and if the polar conic of P cuts PQR in S, then PQ, PS, PR are in harmonic progression; and therefore so are RS, RQ, and RP; and therefore the polar line of R with respect to the polar conic of P passes through Q.

If the coordinates of P, Q, R are $(x_1y_1z_1)$, $(x_2y_2z_2)$, $(x_3y_3z_3)$, we can at once write down the analytical equation which represents this fact, it is

$$x_1x_2x_3+y_1y_2y_3+z_1z_2z_3$$
$$+m\{x_1(y_2z_3+y_3z_2)+y_1(z_2x_3+z_3x_2)+z_1(x_2y_3+x_3y_2)=0.$$

But we have seen before that

$$x_1x_2x_3+y_1y_2y_3+z_1z_2z_3=0,$$

therefore

$$x_1(y_2z_3+y_3z_2)+y_1(z_2x_3+z_3x_2)+z_1(x_2y_3+x_3y_2)=0.$$

An equation which like that first obtained is independent of m.

I proceed now to give a demonstration of this latter equation which will be quite independent of the first, as follows.

The equation whose roots are the various values of

$$\alpha\alpha'+\beta\beta'+\gamma\gamma',$$

where α, β, γ are the roots of

$$a\theta^3+3b\theta^2+3c\theta+d=0,$$

and α', β', γ' those of

$$a'\theta^3 + 3b'\theta^2 + 3c'\theta + d' = 0,$$

is $$\left(z - \frac{3bb'}{aa'}\right)^3 - 27\frac{HH'}{a^2a'^2}\left(z - \frac{3bb'}{aa'}\right) = \frac{27}{2}\left\{\frac{GG' \pm \Delta\Delta'}{a^3a'^3}\right\},$$

where $H = b^2 - ac$, $G = 2b^3 - 3abc + acd$, and $\Delta^2 = G^2 - 4H^3$,
$H' = b'^2 - a'c'$, &c.

Now $$\alpha(\beta' + \gamma') + \beta(\gamma' + \alpha') + \gamma(\alpha' + \beta') = 0,$$

if $$\alpha\alpha' + \beta\beta' + \gamma\gamma' = \frac{9bb'}{aa'},$$

or, if $$16\frac{b^3b'^3}{a^3a'^3} - 12\frac{bb'}{aa'} \cdot \frac{HH'}{a^2a'^2} = \frac{GG' \pm \Delta\Delta'}{a^3a'^3},$$

or $$16b^3b'^3 - 12bb'HH' = GG' \pm \Delta\Delta' \quad (1).$$

Now the values of the ratio x/z for the points $(x_1y_1z_1)$, $(x_2y_2z_2)$, $(x_3y_3z_3)$, where $\lambda x + \mu y + \nu z = 0$ cuts

$$x^3 + y^3 + z^3 + 6mxyz = 0$$

are given by

$$\mu^3(\theta^3 + 1) - (\lambda\theta + \nu)^3 - 6m\mu^2\theta(\lambda\theta + \nu) = 0,$$

or $$\theta^3(\mu^3 - \lambda^3) - 3\theta^2(\lambda^2\nu + 2m\lambda\mu^2) - 3\theta(\lambda\nu^2 + 2m\mu^2\nu) + \mu^3 - \nu^3 = 0,$$

and the values of y/z by

$$\theta^3(\lambda^3 - \mu^3) - 3\theta^2(\mu^2\nu + 2m\lambda^2\mu) - 3(\mu\nu^2 + 2m\lambda^2\nu) + \lambda^3 - \nu^3 = 0,$$

which is got from the preceding by interchanging λ and μ. Now condition (1) is satisfied by these two equations, for

$$\begin{aligned} H &= (\lambda^2\nu + 2m\mu^2\lambda)^2 + (\mu^3 - \lambda^3)(\lambda\nu^2 + 2m\mu^2\nu) \\ &= \mu^2\{4m^2\lambda^2\mu^2 + 2m\nu(\lambda^3 + \mu^3) + \lambda\mu\nu^2\}. \\ &= \mu^2h, \text{ say.} \end{aligned}$$

Similarly $H' = \lambda^2h$, for h remains unchanged when we interchange λ, μ.

Again

$$\begin{aligned} G = &-2\{\lambda^2\nu + 2m\mu^2\lambda\}^3 \\ &- 3(\mu^3 - \lambda^3)(\lambda^2\nu + 2m\mu^2\lambda)(\lambda\nu^2 + 2m\mu^2\nu) + (\mu^3 - \lambda^3)^2(\mu^3 - \nu^3), \end{aligned}$$

therefore

$$\begin{aligned} G = \mu^3\{&-16\lambda^3\mu^3 - 12\lambda\mu\nu(\lambda^3 + \mu^3)m^2 \\ &- 12\lambda^2\mu^2\nu^2m + \lambda^6 + \mu^6 - \nu^3(\lambda^3 + \mu^3) - 2\lambda^3\mu^3\}; \end{aligned}$$

therefore $G = \mu^3g$, say, and similarly $G' = \lambda^3g$.

Again
$$bb' = (\lambda^2\nu + 2m\lambda\mu^2)\,(\mu^2\nu + 2m\lambda^2\mu)$$
$$= \lambda\mu\,\{\lambda\nu + 2m\mu^2\}\,\{\mu\nu + 2m\lambda^2\}$$
$$= \lambda\mu h,$$
$$\Delta^2 = G^2 - 4H^3 = \mu^6\,(g^2 - 4h^3),$$
$$\Delta'^2 = G'^2 - 4H'^3 = \lambda^6\,(g^2 - 4h^3),$$
therefore
$$\Delta\Delta' = \lambda^3\mu^3\,(g^2 - 4h^3);$$
therefore
$$16b^3b'^3 - 12bb'HH'$$
$$= \lambda^3\mu^3\,\{16h^3 - 12h^3\} = 4\lambda^3\mu^3h^3$$
$$= \lambda^3\mu^3\,\{g^2 - (g^2 - 4h^3)\}$$
$$= GG' - \Delta\Delta'.$$

Therefore the condition is satisfied.

We have then
$$\frac{x_1}{z_1}\left\{\frac{y_2}{z_2} + \frac{y_3}{z_3}\right\} + \frac{x_2}{z_2}\left\{\frac{y_3}{z_3} + \frac{y_1}{z_1}\right\} + \frac{x_3}{z_3}\left\{\frac{y_1}{z_1} + \frac{y_2}{z_2}\right\} = 0,$$
or
$$x_1\,(y_2z_3 + y_8z_2) + x_2\,(y_3z_1 + y_1z_3) + x_3\,(y_1z_2 + y_2z_1) = 0,$$
or
$$x_1\,(y_2z_3 + y_3z_2) + y_1\,(z_2x_3 + z_3x_2) + z_1\,(x_2y_3 + x_3y_2) = 0.$$

We have seen that
$$x_1\,(y_2z_3 + y_3z_2) + y_1\,(z_2x_3 + z_3x_2) + z_1\,(x_2y_3 + x_3y_2),$$
where $x_1, y_1, z_1;\ x_2, y_2, z_2;\ x_3, y_3, z_3$ are the points where a line meets
$$U \equiv x^3 + y^3 + z^3 + 6mxyz = 0$$
is zero.

Hence
$$\Delta_1\Delta_2\Delta_3\,\{H + m^2U\}$$
is zero where
$$\Delta_1 \equiv x_1\frac{d}{dx} + y_1\frac{d}{dy} + z_1\frac{d}{dz},$$
$$\Delta_2 \equiv \&\text{c.},$$
$$\Delta_3 \equiv \&\text{c.},$$
and H is the Hessian of U.

This is clearly a covariantive result, and we may write
$$\Delta_1\Delta_2\Delta_3\,(\lambda H + 6\mu H) \equiv 0,$$

where H, U are referred to any axes and the ratio of $\lambda : \mu$ is given by

$$\lambda^4 + 24S\lambda^2\mu^2 + 8T\lambda\mu^3 - 48S^2\mu^4 = 0,$$

Salmon's *Higher Plane Curves*, § 227, writing with Salmon, S, T for the two invariants of the cubic.

Now this equation has four roots, so that we infer

$$\Delta_1\Delta_2\Delta_3 U = 0 \enspace (1),$$

$$\Delta_1\Delta_2\Delta_3 H = 0 \enspace (2).$$

We have already seen this to be true for

$$x^3 + y^3 + z^3 + 6mxyz = 0,$$

and we now infer the result for the most general form of the equation of a cubic.

If we make $x_1 : y_1 : z_1 :: x_2 : y_2 : z_2$, we have the equations

$$\Delta_1^2\Delta_3 U = 0,$$

$$\Delta_1^2\Delta_3 H = 0,$$

which are Dr. Salmon's equations for the tangential of the point (x_1, y_1, z_1).

We now proceed to enunciate a property of any curve of odd order, which is analogous to the geometrical property of a cubic quoted above: it may be stated thus.

If a right line meet a curve of the degree $2n+1$ in the points A_1, A_2, ..., A_{2n+1}, and if the first polar of a point in regard to a curve be briefly called its polar, then, take the polar of A, and in respect to this latter take the polar of A_2, so again, in respect to this new curve, take the polar of A_3, and so on until we at length reach the *line* which is the polar of A_{2n}, then the theorem to be proved is that this line will pass through A_{2n+1}. The proof is not difficult; the necessary condition is that

$$\Delta_1\Delta_2\Delta_3 \ldots \Delta_{2n+1} U = 0,$$

where $$\Delta_k \equiv x_k \frac{d}{dx} + y_k \frac{d}{dy} + z_k \frac{d}{dz},$$

and $U = 0$ is the equation of the given curve.

Now $\Delta_1\Delta_2 \ldots \Delta_{2n+1} U$ is evidently of a covariantive character; and therefore it will not be altered by any linear transformation save as to a power of the modulus of transformation.

We proceed then to choose our axes as conveniently as possible, and to this end we take the line $A_1 \ldots A_{2n+1}$ as the line $z = 0$.

Let $U\equiv(a_0, a_1, a_{2n+1})(x, y)^{2n+1}+z(b_0, \dots, b_{2n})(x, y)^{2n}+\dots;$

Δ_1 becomes $x_1\dfrac{d}{dx}+y_1\dfrac{d}{dy}$; Δ_2, $x_2\dfrac{d}{dx}+y_2\dfrac{d}{dy}$; Δ_3, &c.;

so then our expression is

$$a_0x_1x_2\dots x_{2n+1}+a_2\Sigma x_2\dots x_{2n+1}y_1+a_3\Sigma x_3\dots x_{2n+1}y_1y_2+\text{\&c.},$$

multiplied by $(2n+1)!$, where $y_1:x_1$; $y_2:x_2$; &c., are given by

$$(a_0, a_1, \dots, a_{2n+1})(x, y)^{2n+1}=0,$$

so that $\Sigma\dfrac{y_1}{x_1}=-(2n+1)a_1$, $\Sigma\dfrac{y_1y_2}{x_1x_2}=\dfrac{(2n+1)(2n)}{1.2}a_2$,

and so on.

Substituting these in our expression it vanishes identically.

It is interesting to tabulate the expression for a few curves; thus, for

$$\left.\begin{aligned}3y^3&=ax^3+3bx^2+3cx+d\\3y^2&=ax^3+3bx^2+3cx+d\\3y\ &=ax^3+3bx^2+3cx+d\end{aligned}\right\}\text{ it is, }\begin{aligned}-(y_1+y_2+y_3)&=M,\\-(y_1y_2+y_2y_3+y_3y_1)&=M,\\-y_1y_2y_3&=M,\end{aligned}$$

where

$$M\equiv ax_1x_2x_3+b(x_1x_2+x_2x_3+x_3x_1)+c(x_1+x_2+x_3)+d,$$

whilst for

$$x^{2n+1}\pm y^{2n+1}=1,\text{ it is }x_1x_2\dots x_{2n+1}\pm y_1y_2\dots y_{2n+1}=1.$$

ON THE FOCALS OF A QUADRIC SURFACE.

By Prof. *Cayley*.

In plane geometry the focus of a curve is the node of the circumscribed line-system of the curve and the circular points at infinity; and so in solid geometry the focal of a surface or curve is the nodal line of the circumscribed developable of the surface or curve and the circle at infinity. And as in plane geometry the circular points at infinity may be regarded as an indefinitely thin conic, so in solid geometry the circle at infinity may be regarded as an indefinitely thin quadric surface.

In plane geometry, let it be proposed to find the circumscribed line-system (common tangents) of the two quadrics

$$\frac{x^2}{a}+\frac{y^2}{b}+\frac{z^2}{c}=0,\ \frac{x^2}{a'}+\frac{y^2}{b'}+\frac{z^2}{c'}=0;$$

if a common tangent be

$$\xi x+\eta y+\zeta z=0,$$

then we have

$$a\xi^2+b\eta^2+c\zeta^2=0,$$

$$a'\xi^2+b'\eta^2+c'\zeta^2=0;$$

here writing for shortness

$$f,\,g,\,h=bc'-b'c,\ ca'-c'a,\ ab'-a'b,$$

we have

$$\xi^2:\eta^2:\zeta^2=f:g:h,$$

and thence the tangent is

$$x\sqrt{(f)}\pm y\sqrt{(g)}\pm c\sqrt{(h)}=0,$$

viz. we have thus four tangents, and the rationalised form is of course

$$f^2x^4+g^2y^4+h^2z^4-2ghy^2z^2-2hfz^2x^2-2fgz^2y^2=0.$$

In connexion with the corresponding question in solid geometry, I obtain this equation in a different manner. We investigate the envelope of the line $\xi x+\eta y+\zeta z=0$, considering ξ, η, ζ as parameters connected by the foregoing two equations; by the ordinary process of indeterminate multipliers, we have

$$x+(\lambda a+\mu a')\,\xi=0,\ y+(\lambda b+\mu b')\,\eta=0,\ z+(\lambda c+\mu c')\,\zeta=0,$$

and thence, eliminating ξ, η, ζ, we obtain

$$\frac{x^2}{\lambda a+\mu a'}+\frac{y^2}{\lambda b+\mu b'}+\frac{z^2}{\lambda c+\mu c'}=0,$$

$$\frac{ax^2}{(\lambda a+\mu a')^2}+\frac{by^2}{(\lambda b+\mu b')^2}+\frac{cz^2}{(\lambda c+\mu c')^2}=0,$$

$$\frac{a'x^2}{(\lambda a+\mu a')^2}+\frac{b'y^2}{(\lambda b+\mu b')^2}+\frac{c'z^2}{(\lambda c+\mu c')^2}=0,$$

equations equivalent to two equations, from which (λ,μ) are to be eliminated. The second and third equations are the derived functions of the first equation in regard to λ, μ

respectively; and hence, expressing the first equation in an integral form, the result is

$$\text{Disct. } \{x^2(\lambda b+\mu b')(\lambda c+\mu c')+\&\text{c.}\}=0;$$

viz. this is

$$\{(bc'+b'c)x^2+(ca'+c'a)y^2+(ab'+a'b)z^2\}^2$$
$$-4(bcx^2+cay^2+abz^2)(b'c'x^2+c'a'y^2+a'b'z^2)=0,$$

or developing and reducing we have the foregoing result, the coefficients entering through the combinations

$$f,\ g,\ h=bc'-b'c,\ ca'-c'a,\ ab'-a'b.$$

Writing $c=-1$, also $a'=b'=1$, $c'=0$; the equation of the first conic is

$$\frac{x^2}{a}+\frac{y^2}{b}-z^2=0$$

and that of the second may be replaced by the two equations $x^2+y^2=0$, $z=0$, viz. these give the circular points at infinity: we have $f,\ g,\ h=1,\ -1,\ a-b$, and the equation of the line-system is

$$x^4+2x^2y^2+y^4-2h(x^2-y^2)z^2+h^2z^4=0,$$

or if finally $z=1$, then for the tangents from the circular points at infinity to the quadric

$$\frac{x^2}{a}+\frac{y^2}{b}-1=0,$$

the equation is

$$x^4+2x^2y^2+y^4-2h(x^2-y^2)+h^2=0,$$

where as before $h=a-b$; the four tangents intersect in pairs in the two circular points at infinity, and in four other points which are the foci of the quadric.

Passing now to the problem in solid geometry, we consider the two quadric surfaces

$$\frac{x^2}{a}+\frac{y^2}{b}+\frac{z^2}{c}+\frac{w^2}{d}=0,$$

$$\frac{x^2}{a'}+\frac{y^2}{b'}+\frac{z^2}{c'}+\frac{w^2}{d'}=0;$$

if a common tangent plane hereof be

$$\xi x + \eta y + \zeta z + \omega w = 0,$$

then we have

$$a\xi^2 + b\eta^2 + c\zeta^2 + d\omega^2 = 0,$$

$$a'\xi^2 + b'\eta^2 + c'\zeta^2 + d'\omega^2 = 0,$$

and the circumscribed developable is the envelope of the plane, considering in the equation thereof ξ, η, ζ, ω as variable parameters connected by the last two equations. By what precedes it is at once seen that the resulting equation is

$$\text{Disct. } \{x^2 (b\lambda + b'\mu)(c\lambda + c'\mu)(d\lambda + d'\mu) + \&\text{c.}\} = 0,$$

viz. this is a quadric equation in (x^2, y^2, z^2, w^2), the coefficients $a, b, c, d, a', b', c', d'$ entering therein through the combinations

$$\mathrm{a}, \mathrm{b}, \mathrm{c} = bc' - b'c,\ ca' - c'a,\ ab' - a'b;$$

$$\mathrm{f}, \mathrm{g}, \mathrm{h} = ad' - a'd,\ bd' - b'd,\ cd' - c'd.$$

The developed result is given in my paper "On the developable surfaces which arise from two surfaces of the second order," *Camb. and Dub. Math. Jour.*, t. V. (1850), pp. 45–53, see p. 55 (*Works*, t. 11). We require here for the particular case $d = -1$, $a' = b' = c' = 1$, $d' = 0$, viz. one of the surfaces is taken to be

$$\frac{x^2}{a} + \frac{y^2}{b} + \frac{z^2}{c} - w^2 = 0,$$

and the other the circle at infinity $x^2 + y^2 + z^2 = 0$, $w = 0$; we thus have $\mathrm{a}, \mathrm{b}, \mathrm{c} = 1, 1, 1$; $\mathrm{f}, \mathrm{g}, \mathrm{h} = b - c, c - a, a - b$, or now using the italic letters f, g, h to signify these values, we have $f + g + h = 0$. The equation is

$$\begin{aligned}
&f^2x^8 + g^2y^8 + h^2z^8 + f^2g^2h^2w^8\\
&+ 2g(g-h)y^6z^2 + 2h(h-f)z^6x^2 + 2f(f-g)x^6y^2\\
&- 2h(g-h)y^2z^6 - 2f(h-f)z^2x^6 - 2g(f-g)x^2y^6\\
&+ 2f^2(g-h)x^6w^2 + 2g^2(h-f)y^6w^2 + 2h^2(f-g)z^6w^2\\
&- 2f^2gh(g-h)x^2w^6 - 2fg^2h(h-f)y^2w^6 - 2fgh^2(f-g)z^2w^6\\
&+ f^2(f^2-6gh)x^4w^4 + g^2(g^2-6hf)y^4w^4 + h^2(h^2-6fg)z^4w^4\\
&+ (f^2-6gh)y^4z^4 + (g^2-6hf)z^4x^4 + (h^2-6fg)x^4y^4\\
&+ 2gh(gh-3f^2)w^4y^2z^2 + 2hf(hf-3g^2)w^4z^2x^2 + 2fg(fg-3h^2)w^4x^2y^2\\
&+ 2h(gh-3f^2)z^4x^2w^2 + 2f(hf-3g^2)x^4y^2w^2 + 2g(fg-3h^2)y^4z^2w^2\\
&- 2g(gh-3f^2)y^4x^2w^2 - 2h(hf-3g^2)z^4y^2w^2 - 2f(fg-3h^2)x^4z^2w^2\\
&+ 2(gh-3f^2)x^4y^2z^2 - 2(hf-3g^2)x^2y^4z^2 - 2(fg-3h^2)x^2y^2z^4\\
&+ 2(g-h)(h-f)(f-g)x^2y^2z^2\omega^2 = 0.
\end{aligned}$$

The equation may be written in the following four forms:

$$x^2\Theta_1+(gy^2-hz^2+ghw^2)^2\{y^4+2y^2z^2+z^4-2f(y^2-z^2)w^2+f^2w^4\}=0,$$

$$y^2\Theta_1+(hz^2-fx^2+hfw^2)^2\{z^4+2z^2x^2+x^4-2g(z^2-x^2)w^2+g^2w^4\}=0,$$

$$z^2\Theta_1+(fx^2-gy^2+fgw^2)^2\{x^4+2x^2y^2+y^4-2h(x^2-y^2)w^2+h^2w^4\}=0,$$

$$\omega^2\Theta+(x^2+y^2+z^2)^2 \times\{f^2x^4+g^2y^4+h^2z^4-2ghy^2z^2-2hfz^2x^2-2fgx^2y^2\}=0;$$

the last of these shows that the circle at infinity $x^2+y^2+z^2=0$, $w=0$ is a nodal line on the surface; this, however, is not regarded as a focal. The other three show that we have also, as nodal lines, three conics, which are the focals of the given surface, viz. now writing $w=1$, we have for the quadric surface

$$\frac{x^2}{a}+\frac{y^2}{b}+\frac{z^2}{c}-1=0,$$

the three focal conics

$$\begin{array}{lcccl} x=0, & \cdot & -\dfrac{y^2}{h} & +\dfrac{z^2}{g} & -1=0, \\ y=0, & \dfrac{x^2}{h} & \cdot & -\dfrac{z^2}{f} & -1=0, \\ z=0, & -\dfrac{x^2}{g} & +\dfrac{y^2}{gf} & \cdot & -1=0, \end{array}$$

where, as before, $f, g, h=b-c, c-a, a-b$ respectively. If, as usual, a, b, c are positive and in order of decreasing magnitude, then for the ellipsoid

$$\frac{x^2}{a}+\frac{y^2}{b}+\frac{z^2}{c}-1=0$$

we have the focal conics

$$\begin{array}{lcccl} x=0, & \cdot & -\dfrac{y^2}{a-b} & +\dfrac{z^2}{a-c} & -1=0, \\ y=0, & \dfrac{x^2}{a-b} & \cdot & -\dfrac{z^2}{b-c} & -1=0, \\ z=0, & \dfrac{x^2}{a-c} & +\dfrac{y^2}{b-c} & \cdot & -1=0, \end{array}$$

viz. these are an imaginary conic, the focal hyperbola and the focal ellipse in the coordinate planes $x=0$, $y=0$, $z=0$ respectively.

SOLUTION OF $(a, b, \ldots, c) = (a^p, b^p, \ldots, c^p)$.

By *H. W. Lloyd Tanner, M.A.*

1. In the *Messenger* (Vol. XV., pp. 59–61) Prof. Cayley has solved the problem "to find four quantities (no one of them zero) which are in some order or other equal to their squares,

$$(a, b, c, d) = (a^2, b^2, c^2, d^2)."$$

The present paper relates to the same problem when there are n quantities which are in some order or other equal to their pth powers. Here n and p are positive integers, but are otherwise unrestricted; in particular it is not assumed that either of them is prime. Zero values for $a, b, \ldots, c$ are admitted, mainly for the sake of comparison with a passage in Gauss (Art. 9).

2. Suppose that a particular solution is before us. Under each term of the solution write its pth power. The new row will, by hypothesis, consist of the terms of the original row arranged, it may be, in a different order. It follows, as in the theory of substitutions, that the terms can be divided into one or more independent cycles. In any one cycle the terms are arranged in a definite order, viz., each term is the pth power of its predecessor. The partition into cycles is unique, unless some of the terms happen to be equal. We may avoid ambiguity from the existence of equal terms by agreeing that no cycle is to contain equal terms or, in other words, that each cycle is to be closed at the first opportunity. Suppose for example that, p being 3, a solution contains four terms

$$i, \quad i, \; -i, \; -i, \quad (i^2 = -1).$$

Here the second row is

$$-i, \; -i, \quad i, \quad i;$$

and the four terms constitute two cycles

$$(i, -i), \quad (i, -i),$$

not the one cycle $\qquad (i, -i, i, -i)$.

3. We consider first (Arts. 3–6) those solutions which consist of a single n-nomial cycle. If a is one of the terms, the cycle is

$$a, \; a^p, \; a^{p^2}, \; \ldots, \; a^{p^{n-1}},$$

with the conditions (i) that

$$a^{p^n} = a \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (N),$$

and (ii) that n is the least number for which (N) is true.

Conversely, if these conditions are satisfied, a is a term of an n-nomial cycle.

It follows that a may be zero when $n = 1$, but not when n has any greater value. The other values of a are roots of unity. Suppose that one of these values, a, is a root of order ω; that is to say, a^ω is the lowest power of a which is unity. Then, to satisfy the condition (i), ω must be a factor of $p^n - 1$; and, to satisfy the condition (ii), ω must not be a factor of any one of the quantities

$$p - 1,\ p^2 - 1,\ p^3 - 1,\ \ldots,\ p^{n-1} - 1,$$

or, what is equivalent, ω must not be a factor of any one of the quantities

$$p^\alpha - 1,\ p^\beta - 1,\ \ldots,\ p^\gamma - 1;$$

where the exponents $\alpha, \beta, \ldots, \gamma$ are all the factors of n, such that n/α, n/β, ..., n/γ are primes. [For example, $n = 24$, it is the same thing to say that ω (a factor of $p^{24} - 1$) must not be a factor of either

$$p^8 - 1 \text{ or } p^{12} - 1,$$

as to say that it must not be factor of either

$$p - 1,\ p^2 - 1,\ p^3 - 1,\ \ldots, \text{ or } p^{23} - 1].$$

4. The number of different n-nomial cycles will be denoted by the symbol (n).

Then $$(1) = p,$$

for a may be any root of the equation

$$x^p - x = 0,$$

and all these roots are different.

When n is a prime, the n-nomial cycles are made up of terms which satisfy the equation

$$x^{p^n} - x = 0,$$

but not $$x^p - x = 0;$$

that is to say, the terms are all the roots of

$$(x^{p^n} - x)/(x^p - x) = 0.$$

Thus there are $p^n - p$ terms, none of them equal.

On the other hand, each of the (n) cycles contains n terms and two different cycles cannot have a term in common. Thus the number of terms is

$$n \times (n).$$

Equating these values, we find that, when n is a prime,

$$n \times (n) = p^n - p.$$

When n is any number, the formula becomes

$$n \times (n) = p^n - \Sigma p^{n_1} + \Sigma p^{n_2} - \&\text{c.} \ldots\ldots\ldots\ldots (\text{I}),$$

where $n_1, n_2, \ldots, n_r$ are factors of n, such that n/n_r is the product of r different prime factors, each in the first power only. Of course r is taken to be 1, 2, 3, &c., in succession. The Σ's on the right of (I) cover all the terms that satisfy the condition.

The left-hand side of (I) is the number of terms a, which appear in the (n) n-nomial cycles. That the other side is the same number, we prove by writing down the equation satisfied by the terms a of the n-nomial cycles, and by these only. It will be found that this equation has no equal roots and its degree is the dexter of (I).

To simplify the expression, I write ΠN_r to represent the product of all the functions

$$x^{p^{n_r}} - x,$$

for different values of n_r, but the same r. The equation in question, with this notation, is

$$(N . \Pi N_2 . \Pi N_4 . \ldots) / (\Pi N_1 . \Pi N_3 . \ldots) = 0 \ldots\ldots (\mathfrak{N}).$$

It is at once seen that this equation is of the degree indicated by the right-hand side of (I). To prove that it has the proper roots, observe that any particular N_r, say

$$x^{p^\nu} - x = 0,$$

where ν is one of the values of n_r, has all its roots different, so that, if $x - b$ is a k-fold factor of ΠN_r, it is a factor of k different expressions, N_r. Further, all the roots of

$$x^{p^\nu} - x = 0$$

are roots of $\qquad x^{p^\mu} - x = 0,$

when μ is any multiple of ν. In particular they all are included in the roots of

$$N, = x^{p^n} - x, = 0.$$

The factors of N may then be divided into two classes, viz. factors $x - a$, which are factors of N only; and factors $x - b$, which are common to N and some of the N_r. The former set persist unchanged in $\mathfrak{N}$; the latter disappear, as we shall prove.

Suppose a particular factor, $x - b$, occurs k times in ΠN_1. (This implies that there are at least k different expressions N_1 and therefore that n contains at least k different factors, so that there is at least one N_k). Then $(x - b)$ occurs in ΠN_2 raised to the power $k(k-1)/2\,!$; in ΠN_3 raised to the power $k(k-1)\,k-2)/3\,!$; and so on, until finally it appears once in the product ΠN_k. Hence, upon the whole, it occurs in $\mathfrak{N}$ raised to the power

$$1 - k + \frac{k(k-1)}{2\,!} - \ldots + (-1)^k, = 0,$$

that is to say, it disappears from $\mathfrak{N}$.

5. This discussion is, it will be noticed, essentially a replica of the theory of the primitive roots of

$$x^n - 1 = 0.$$

The equation for the primitive roots is of the form $\mathfrak{N}$, with n written for p^n and n_r for p^{n_r}, and the degree of this equation is

$$\tau n, = n - \Sigma n_1 + \Sigma n_2 - \&c.,^*$$

corresponding to (I).

The verification of the equation for primitive roots depends on the theorem that all the roots common to the equations

$$x^m - x^0 = 0, \quad x^n - x^0 = 0,$$

are all the roots of $\qquad x^g - x^0 = 0,$

where g is the greatest common measure of m, n. In like manner the verification of $\mathfrak{N}$ depends on the theorem that all the roots common to

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* Prof. Sylvester calls the numbers, which are prime to n and not greater than n, the *totitives* of n. The number of the totitives of n he calls the *totient* of n, and indicates this number by the symbol τn. The convenience of having a nomenclature and a cognate notation will, I hope, justify the use of τn for the familiar ϕn.

$$x^{p^m} - x^{p^0} = 0, \quad x^{p^n} - x^{p^0} = 0,$$

are all the roots of $\qquad x^{p^g} - x^{p^0} = 0,$

where g is the greatest common measure of m, n. And it is interesting to observe that the proof of this last theorem depends upon the truth of its analogue. For any root of any of the three equations last written must be either 0 (which is obviously common to all) or a root of unity, say of order ω. If this unit-root is common to the pair, ω is any common factor of $p^m - p^0$ and $p^n - p^0$; if it belongs to the last equation, ω is any factor of $p^g - p^0$. But the two systems of values of ω thus determined are identical.

The analogy between the two theories serves to suggest another theorem which is required in the sequel. It corresponds to the well-known theorem

$$n = \tau n + \tau\alpha + \tau\beta + \ldots + \tau 1,$$

where $\alpha, \beta, \ldots, 1$ are all the divisors of n. It is

$$p^n = n\,(n) + \alpha\,(\alpha) + \beta\,(\beta) + \ldots + 1\,(1) \;\ldots\ldots\text{(II)}.$$

The left side of (II) is the number of the roots of

$$x^{p^n} - x = 0.$$

The right side enumerates the same roots arranged in sets. The first set comprises the roots that occur in n-nomial cycles (and in no lower cycles); the second those that occur in the α-nomial cycles (and in no lower cycles); and so on. It is clear that this second enumeration is simplex and exhaustive.

6. The following table exhibits the cycles for $p = 3$, $n = 1$, 2, 3, 4, 5, and will serve to illustrate the preceding analysis. The first two columns contain n and $p^n - 1$ respectively. The third column contains ω, a factor of $p^n - 1$, which is not a factor of any similar expression with a lower value of n. The form of the cycle is shown in the fourth column, where a represents any root of order ω. By putting for a all its values we obtain all the cycles. These can all be expressed in terms of a; for instance, $\omega = 13$, the four cycles are

$$a,\ a^3,\ a^9;\quad a^2,\ a^6,\ a^5;\quad a^4,\ a^{12},\ a^{10};\quad a^8,\ a^{11},\ a^7.$$

But it seemed sufficient for the present purpose merely to give the general form (a, a^3, a^9) and the number of cycles included in it. This number is shown in column 5. In the last column is a numerical verification of the equation (I). The value of (n) is found by adding the numbers in column 5 which belong to n.

I may remark that the fact that all the entries in column 5 are necessarily positive integers is a theorem of some moment in the theory of the divisors of cyclotomic functions.

n	$p^n - 1$	ω	Cycle.	Number of Cycles.	$(n) = \frac{1}{n}(p^n - \&c.)$
1			0	1	
.	2	1	1	1	
.		2	-1	1	$3 = 3$
2	8	4	$i, -i$	1	
.		8	a, a^3	2	$3 = \frac{1}{2}(3^2 - 3)$
3	26	13	a, a^3, a^9	4	
.		26	a, a^3, a^9	4	$8 = \frac{1}{3}(3^3 - 3)$
4	80	5	a, a^3, a^4, a^2	1	
.		10	a, a^3, a^9, a^7	1	
.		16	a, a^3, a^9, a^{11}	2	
.		20	a, a^3, a^9, a^7	2	
.		40	a, a^3, a^9, a^{27}	4	
.		80	a, a^3, a^9, a^{27}	8	$18 = \frac{1}{4}(3^4 - 3^2)$
5	242	11	a, a^3, a^9, a^5, a^4	2	
.		22	a, a^3, a^9, a^5, a^{15}	2	
.		121	$a, a^3, a^9, a^{27}, a^{81}$	22	
.		242	$a, a^3, a^9, a^{27}, a^{81}$	22	$48 = \frac{1}{5}(3^5 - 3)$

7. We now pass to the general case in which the n terms of a solution are not assumed to belong to one cycle. Let the number of r-nomial cycles $(r = 1, 2, 3, \ldots)$ in a solution be denoted by k_r. Then in order that the number of terms may be right we must have

$$n = k_1 + 2k_2 + 3k_3 + \ldots + rk_r + \ldots \quad \ldots\ldots\ldots (K).$$

Now the k_r cycles are selected from (r) cycles, all different; and in the selection repetitions are admitted. Hence the number of different selections that can be made is the coefficient of x^{rk_r} in the expansion of

$$(1-x^r)^{-(r)}.$$

The product of these numbers for $k_1, k_2, k_3, \ldots$ is then the number of different solutions for this particular set of values of the k. If another set of values of k be taken the new solutions will all be different from those already counted. Hence the number of solutions is found by taking the sum of the products just mentioned for every set of values of the k consistent with the condition (K). But this is just the way in which we form the coefficient of x^n in the expansion of

$$(1-x)^{-(1)}(1-x^2)^{-(2)}(1-x^3)^{-(3)}\ldots\ldots$$

and this expression is therefore a generating function for the number of solutions.

Now Gauss has proved that

$$(1-x)^{-(1)}(1-x^2)^{-(2)}(1-x^3)^{-(3)}\ldots = (1-px)^{-1}\ldots\text{(III)},$$

by differentiating the logarithm of each side with respect to $\log x$, expanding the several fractions that result, and comparing the two coefficients of x^n. On the left the coefficient is

$$(1) + \alpha(\alpha) + \beta(\beta) + \ldots + n(n),$$

where $1, \alpha, \beta, \ldots$ are all the divisors of n. On the right it is p^n; and these are equal by reason of (II). The equality of the differentiated expressions shows that the two sides of (III) are in a constant ratio; and that this ratio is 1 appears at once.

The number of different solutions in the general case is therefore p^n.

It is suggestive that this number should be the same as the number of the different terms that can be present in the solutions; but I do not see any correspondence between the individuals of the two sets.

8. If we exclude zero values of $a, b, \ldots, c$ the number of solutions is $p^n - p^{n-1}$.

For from every n-nomial solution containing at least one 0 we can derive an $(n-1)$-nomial solution by merely omitting a 0. These are all different, and they make up the complete set of $(n-1)$-nomial solutions; since by adding a 0 to any $(n-1)$-nomial solution we obtain an n-nomial solution. Hence the number of n-nomial solutions with at least one zero is p^{n-1}; and therefore as stated the number of solutions containing n elements, none of them zero, is

$$p^n - p^{n-1}.$$

9. The theorem (III) quoted from Gauss in Art. 7 is given in the unfinished *Disquisitiones generales de congruentiis* (*Werke*, Bd. II., p. 221) and it occurs in the course of an investigation (pp. 219–222) into the number (n) of different integral rational functions of the nth degree which are irreducible with respect to a given prime modulus p. Gauss observes that the number of different functions (reducible and irreducible) of the nth degree is p^n. Hence he obtains (n) in particular cases, viz. (1), (2) up to (8). From these in turn he derives the particular cases of (II) which answer to $n = 1, 2, 3, \ldots, 8$, and the general form of (II) thus suggested is verified by means of (III) in which (1), (2) ... (n) are regarded as the unknown quantities to de determined. In the following article (No. 345) the theorem (II) is modified in expression, and Gauss adds "Facile nunc est ex hoc theoremate valorem expressionis (n) ipsum deducere," and gives the general form equivalent to (I).

In the problem of the present paper we can prove, as in Prof. Cayley's paper (*loc. cit.*), that there are p^n solutions, but it is not obvions that these solutions are all different. I have thought it desirable therefore to change the order and so avoid the necessity of assuming the number of different solutions to be p^n.*

10. The n terms of any solution may be regarded as roots of an equation of the nth degree,

$$Fx = 0,$$

where

$$Fx = (x-a)(x-b)\ldots(x-c)$$

$$= x^n - Px^{n-1} + Qx^{n-2} - \&\text{c.}, \quad \text{(say)},$$

and then we may seek the values of P, Q, &c.

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* Expressions similar to the dexter of (I) occur in a paper on Cyclotomic Functions which I recently communicated to the London Mathematical Society (*Proc.*, Vol. XX., pp. 281, 282). Let $s_1, s_2, s_3, \ldots$ represent, as usual, the sums of the 1st, 2nd, 3rd, ... powers of the roots of an equation, E, in which the highest power of x has the coefficient 1. In (I) write $s_1, s_2, s_3, \ldots$ instead of $p, p^2, p^3, \ldots$; then the necessary and sufficient conditions that the coefficients of E should be integral are that (1), (2), (3), &c., should all be integers.

Let α represent a primitive pth root of unity. Then

$$\begin{aligned} Fx \, . \, F(x\alpha) \, . \, F(x\alpha^2) \ldots F(x\alpha^{p-1}) &= (x^p - a^p)(x^p - b^p) \ldots (x^p - c^p) \\ &= (x^p - a)(x^p - b) \ldots (x^p - c) \\ &= F(x^p). \end{aligned}$$

Equating the coefficients of the same powers of x in the first and last terms of this chain, we obtain n equations each of degree p in P, Q, &c. This system has p^n solutions which, as we have seen, are all different. The P-equation of the system (namely that formed by eliminating the other coefficients, Q, &c. of Fx) can without much difficulty be obtained in the form of a product of irreducible factors.

The factors corresponding to solutions which consist of single n-nomial cycles can be written down at once from Reuschle's Tables. For in these cases

$$P, = a + a^p + a^{p^2} + \ldots + a^{p^{n-1}},$$

is a Gaussian period; so that P satisfies the cyclotomic equation for the n-nomial periods of roots of order ϖ.

Thus, if $p = 3$, $n = 3$ and $\varpi = 13$ (see the table of Art. 6) the corresponding factor of the P-equation is

$$P^4 + P^3 + 2P^2 - 4P + 3, = \chi(P),$$

the cyclotomic for the trinomial periods of the 13th roots of 1.

Moreover, if P be any value which nullifies $\chi(P)$ the three terms of the corresponding period are given by an equation, one form of which is

$$\psi a =, a^3 - Pa^2 - \{(P-2)/(P+1)\} a - 1 = 0 \ldots (\alpha).$$

If a solution consists of more than one cycle, the corresponding P is the sum of two or more periods each of which satisfies a cyclotomic equation. The cognate factor of the P-equation is formed by an easy elimination. Whether the factor is irreducible or not depends upon the character of the periods in question; for instance, if the periods are all the same the factor is obviously a power of the cyclotomic.

11. I conclude this paper with a brief reference to the related congruence problem,

$$(a, b, \ldots, c) \equiv (a^p, b^p, \ldots, c^p), \text{ (mod. } \mu).$$

The following table gives the cycles in the solution when $p = 3$, $\mu = 313$. The three columns here given answer to the first, third and fourth columns of the table in Art. 6. The cycles are given in full except the 6-nomial cycles which are omitted to economize space—

n	ω	Cycles
1	.	0
.	1	1
.	2	-1
2	4	$25, -25$
.	8	$5, 125 : -5, -125$
3	13	$-19, 27, -36 : -33, 58, 113 : 44, 48, 103 : -64, 150, -79$
.	26	$19, -27, 36 : 33, -58, -113 : -44 -48 -103 : 64 -150, 79$
6	52	four cycles
.	104	eight cycles.

It will be seen that so far as it goes this table corresponds very closely with that of Art. 6. But there are here no. 4- or 5-nominal cycles (because to modulus 313 there are no roots of unity of order 5 or 11): in fact the table here given exhausts the cycles. The equations $\chi P=0$, $\psi(a)=0$ apply to the congruence problem. For example, the roots of

$$\chi P = P^4 + P^3 + 2P^2 - 4P + 3 \equiv 0, \text{ (mod. 313)},$$

are
$$-28, 138, -118, 7,$$

and these are the sums of the four cycles belonging to $\omega=13$.

Selecting any one of these, say -28, we have

$$\psi a =, a^3 + 28a^2 + 138a - 1 \equiv 0, \text{ (mod. 313)},$$

$$\{\text{for } -(P-2)/(P+1) = -30/27 \equiv 138, \text{ (mod. 313)}\}.$$

It is easily verified that the three terms

$$-19, 27, -36$$

satisfy this congruence.

In the instances I have tried it has been easier to form the cycles directly from the ωth roots of unity, (mod. μ), than indirectly by using the congruences such as those shown above.

12. It has been seen that in the solution of the congruence problem there may be no cycles corresponding to a selected cycle in the solutions of the equation problem. This tends to make the (1), (2), (3), ... less for the congruence than they

are for the equation. When the modulus is prime this reduction is uncompensated. But when the modulus is composite there may be a change in the opposite direction, one cycle in the equation problem corresponding to several in the congruence problem. This always happens, for example, with one-termed cycles when p is odd and μ contains more than one odd prime. Examples will be found by examining the generating functions written below. The number of n-nomial solutions is in each case the coefficient of x^n; the number of r-nomial cycles is the index (with changed sign) of the factor $1-x^r$. Throughout, $p=3$.

$$\begin{array}{rll}
\mu = & 17, & \text{G.F.} = (1-x)^{-3}(1-x^2)^{-3}(1-x^4)^{-2}, \\
& 313, & \phantom{\text{G.F.} = }(1-x)^{-3}(1-x^2)^{-3}(1-x^3)^{-8}(1-x^6)^{-12}, \\
& 21, & \phantom{\text{G.F.} = }(1-x)^{-5}, \\
& 45, & \phantom{\text{G.F.} = }(1-x)^{-5}(1-x^2)^{-2}.
\end{array}$$

A convenient means of detecting any omission of cycles is furnished by the following theorem, in which (1), (2), ... represent the numbers of different cycles with 1, 2, ... terms respectively. It will be understood that they are different from the (1), (2), ... of Art. 4.

Remove from $\tau\mu$ by division every factor it has in common with p. The quotient increased by 1 will be equal to

$$(1) + 2\,(2) + 3\,(3) + \ldots .$$

These are in fact two ways of expressing the number of the terms in all the cycles. The p-factors are removed from $\tau\mu$ because ω must be prime to p. The reduced $\tau\mu$ counts all the available roots of unity, (mod. μ), but omits 0. Hence it is to be increased by 1 when zero values are admitted. The following verifications of the G.F.'s given above will suffice to illustrate:

$$\begin{array}{rlrl}
\mu = & 17, \ \tau\mu = & 16, & 16 + 1 = 1.3 + 2.3 + 4.2, \\
& 313, & 312, & 104 + 1 = 1.3 + 2.3 + 3.8 + 6.12, \\
& 21, & 12, & 4 + 1 = 1.5, \\
& 45, & 24, & 8 + 1 = 1.5 + 2.2.
\end{array}$$

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NOTE ON CLAPEYRON'S THEOREM OF THE THREE MOMENTS.

By *Karl Pearson.*

(1) Clapeyron in a paper published in the *Comptes rendus*, T. xlv., 1857, pp. 1076-80, has shown that the following relation holds between the moments M_3, M_4, M_5 at three successive points of support of a continuous beam, the spans between these points of support being of length l_{34} and l_{45} respectively, and there being continuous uniform loads of p_{34} and p_{45} per unit run on the two spans:

$$l_{34}M_3 + 2(l_{34} + l_{45})M_4 + l_{45}M_5 = \tfrac{1}{4}(p_{34}l_{34}^3 + p_{45}l_{45}^3) \text{ ......(i).}$$

Heppel in a memoir published in Vol. xix (1859-60), pp. 625-43, of the *Proceedings of the Institution of Civil Engineers*, shows that $\frac{3}{8}(W_{34}l_{34}^2 + W_{45}l_{45}^2)$ must be added to the right-hand side of (i) if there be isolated central loads W_{34} and W_{45} on the spans l_{34}, l_{45} respectively. He further shows that the reaction R_4 at the fourth point of support is given by

$$R_4 = \tfrac{1}{2}(p_{34}l_{34} + W_{34} + p_{45}l_{45} + W_{45}) + \frac{M_4 - M_3}{l_{34}} + \frac{M_4 - M_5}{l_{45}} \text{ ....(ii).}$$

Weyrauch has generalised these results in his *Allgemeine Theorie und Berechnung der continuirlichen und einfachen Träger*, 1873, p. 8. He supposes any system of isolated loads and the points of support not necessarily on the same level. If f_4 be the depth of the fourth point of support below a given horizontal line, then the following term must be added to the left-hand side of (i), $E\omega k^2$ being the flexural rigidity of the continuous beam:

$$6E\omega k^2 \left\{\frac{f_3 - f_4}{l_{34}} + \frac{f_4 - f_4}{l_{45}}\right\}.$$

For our present purposes we may accordingly write these results in the form:

$$l_{34}M_3 + 2(l_{34} + l_{45})M_4 + l_{45}M_5 + 6E\omega k^2 \left\{\frac{f_3 - f_4}{l_{34}} + \frac{f_5 - f_4}{l_{45}}\right\}$$
$$= \tfrac{1}{4}(p_{34}l_{34}^3 + p_{45}l_{45}^3) + \tfrac{3}{8}(W_{34}l_{34}^2 + W_{45}l_{45}^2) \text{.........(iii).}$$

For R_4 we have the value given by (ii).

(2) Now the above extended form of Clapeyron's Theorem is based by its investigator on the assumption that the points of support are *rigid.* But in most practical cases these 'points of support' are metal columns as elastic as the continuous girder itself, and it seems worth while investigating whether this fact may not sensibly influence the values of the reactions as calculated from the equations (i), (ii), and (iii) above. Such is the object of the present note.

Dorna in a memoir published in *Serie* II., T. XVIII., pp. 281-8, of the *Memorie dell' Accad. delle Scienze di Torino* (1857), has gone to the opposite extreme and supposed the supported body rigid and the supports elastic. He has been followed by Ménebréa in a memoir of 1858 (*Comptes rendus*, XLVI., pp. 1056-60). Which of these two assumptions is really more approximate in practise, or is it necessary to reject both of them?

That the question is not without importance may be seen from the following case. Suppose a beam uniformly loaded with a total load W to rest on three columns of equal height and of cross-sections ω_1, ω_2, ω_1, and stretch-moduli E_1, E_2, E_1 respectively. Then Dorna's theory gives for the three reactions

$$R_1 = R_3 = \frac{E_1\omega_1}{2E_1\omega_1 + E_2\omega_2} W, \quad R_2 = \frac{E_2\omega_2}{2E_1\omega_1 + E_2\omega_2} W.$$

Supposing the supports rigid, we deduce with Clapeyron

$$R_1 = R_3 = \tfrac{3}{16} W, \quad R_2 = \tfrac{5}{8} W$$

The two theories would only agree if $\dfrac{E_1\omega_1}{E_2\omega_2} = \tfrac{3}{10}$, or since, generally, $E_1 = E_2$, if the central column had a cross-section $\tfrac{10}{3}$ that of either terminal column. Thus they lead as a rule to very divergent values for the reactions.

Furthermore, it is easy to show that a very slight change in the height of one of the columns does on Clapeyron's hypothesis make a great difference in the values of the reactions. For example, in the above case, let the middle column be lower than the terminal columns by $1/n$ the deflection of the beam when the middle point of support is removed, then we find (Cotterill's *Applied Mechanics* p. 331):

$$R = \tfrac{5}{8} W\left(1 - \frac{1}{n}\right).$$

Hence it would appear that a small change in the height of one of the columns would seriously influence the reactions, and such changes of height might really be produced by the columns being elastic.

(3) Let the height of the rth column be h_r, lets its stretch-modulus be E_r, and its cross-section α_r, and let us suppose the tops of all the columns at the same horizontal level before the girder is placed upon them, their differences of height depending on the sub-structure.

Then we have

$$R_r = E_r \alpha_r f_r / h_r \dots\dots\dots\dots\dots\dots\text{(iv)}.$$

We can now easily generalise Clapeyron's theorem. Equations (iv) and (ii) give us f_r in terms of the moments, and this substituted in (iii) gives us a relation between the moments to replace Clapeyron's Theorem. It takes the following form:

$$\begin{aligned}
&l_{34}M_3 + 2\,(l_{34}+l_{45})\,M_4 + l_{45}M_5 \\
&\quad + 6\left[\frac{k^2}{l_{23}l_{34}}\frac{h_3}{n_3}(M_3 - M_2) + \frac{k^2}{l_{34}^{\ 2}}\left\{\frac{h_3}{n_3} + \frac{h_4}{n_4}\left(1+\frac{l_{34}}{l_{45}}\right)\right\}(M_3 - M_4)\right. \\
&\quad \left. + \frac{k^2}{l_{45}^{\ 2}}\left\{\frac{h_5}{n_5} + \frac{h_4}{n_4}\left(1+\frac{l_{45}}{l_{34}}\right)\right\}(M_5 - M_4) + \frac{k^2}{l_{45}l_{56}}\frac{h_5}{n_5}(M_5 - M_6)\right] \\
&= \tfrac{1}{4}\,(p_{34}l_{34}^{\ 3} + p_{45}l_{45}^{\ 3}) + \tfrac{3}{8}\,(W_{34}l_{34}^{\ 2} + W_{45}l_{45}^{\ 2}) - \frac{3k^2}{l_{34}}\frac{h_3}{n_3}(w_{23} + w_{34}) \\
&\quad + 3\left(\frac{k^2}{l_{34}} + \frac{k^2}{l_{45}}\right)\frac{h_4}{n_4}(w_{34} + w_{45}) - \frac{3k^2}{l_{45}}\frac{h_4}{n_5}(w_{45} + w_{56}) \dots\dots \text{(v)},
\end{aligned}$$

where w_r = total load on rth span and $n_r = E_r\alpha_r / (E\omega)$.

This equation involves *five* moments, and so may be spoken of as the *theorem of the five moments.* I have written it down in the most general form that is likely to occur in any practical calculations of continuous beams; as a rule it will be simpler in form.

(4) Obviously the order of terms introduced by supposing the columns elastic is given by

$$\frac{k^2}{l_r^2}\,\frac{h_r}{l_r}\,\frac{1}{n_r}.$$

To get some idea of the value of this we note that n_r will possibly not be very different from unity, h_r/l_r will also hardly be likely to exceed this limit. Hence we may say that the terms introduced are of the order $\left(\frac{k}{l_r}\right)^2$, and in most practical cases this would be small. Hence to a first approximation the support moments are given by the theorem of the three moments. Let their values so calculated be given by N_r, and suppose $M_r = N_r + z_r$, z_r being small, then we have to determine the successive additions to the moments N_r, also a three moment theorem expressed by the equation:

$$l_{34}z_3 + 2(l_{34} + l_{45})z_4 + l_{45}z_3 = 6\left[\frac{k^2}{l_{23}l_{34}}\frac{h_3}{n_3}(N_2 - N_3)\right.$$

$$+ \frac{k^2}{l_{34}^{\ 2}}\left\{\frac{h_3}{n_3} + \frac{h_4}{n_4}\left(1 + \frac{l_{34}}{l_{45}}\right)\right\}(N_4 - N_3)$$

$$+ \frac{k^2}{l_{45}^{\ 2}}\left\{\frac{h_5}{n_5} + \frac{h_4}{n_4}\left(1 + \frac{l_{45}}{l_{34}}\right)\right\}(N_4 - N_5)$$

$$\left. + \frac{k^2}{l_{45}l_{56}}\frac{h_5}{n_5}(N_6 - N_5)\right] - \frac{3k^2}{l_{34}}\frac{h_3}{n_3}(w_{23} + w_{34})$$

$$+ 3\left(\frac{k^2}{l_{34}} + \frac{k^2}{l_{45}}\right)\frac{h_4}{n_4}(w_{34} + w_{45}) - \frac{3k^2}{l_{45}}\frac{h_5}{n_5}(w_{45} + w_{56}) \ldots\ldots \text{(vi)}.$$

The right-hand side of this equation consisting entirely of known quantities, we have a finite difference equation of the third order as in Clapeyron's theorem to solve.

(5) Returning to equation (v), let us ask when it will reduce to Clapeyron's three moments theorem. Obviously, if we suppose $f_1 = f_2 = f_3 = \ldots = f_r = \ldots$. That is to say, the tops of the columns in the same horizontal after strain. But by (iv) this involves

$$\frac{R_1 h_1}{E_1 \alpha_1} = \frac{R_2 h_2}{E_2 \alpha_2} = \ldots = \frac{R_r h_r}{E_r \alpha_r} = \ldots\ldots\ldots\ldots \text{(vii)}.$$

Hence: *if the reactions be calculated by the theorem of the three moments and the cross-sections of the columns be then found by* (vii), *we obtain accurate results, although the supports are supposed elastic and not rigid by this theorem.*

For the case of equal heights and equal elasticity in the columns, we have

$$\frac{R_1}{\alpha_1} = \frac{R_2}{\alpha_2} = \ldots = \frac{R_r}{\alpha_r} = \ldots \ \ldots\ldots\ldots\ldots \text{(viii)},$$

or the stress will be equal in all the columns, a distinct practical advantage.

(6) Suppose all the columns of the same form and material and all the spans equal, then (v) becomes

$$M_3 + 4M_4 + M_5 - \frac{6k^2h}{nl^3}\{4(M_3 + M_5) - 6M_4 - M_2 - M_6\}$$

$$= \tfrac{1}{4}l^2(p_{34} + p_{45}) + \tfrac{3}{8}l(W_{34} + W_{45})$$

$$- \frac{3k^2h}{nl^2}(w_{23} + w_{56} - w_{34} - w_{45}) \text{ ...... (ix).}$$

Let us apply this to the case of the uniform girder weighing W tons and resting on one central and two terminal columns of equal height, cross-section and material. We have all the M's zero except M_4 and $2lp_{34} = 2lp_{45} = W$, $w_{23} = w_{56} = 0$, and $w_{34} = w_{45} = W/2$. Hence, if $\beta = \dfrac{k^2h}{nl^3}$, we find

$$4M_4(1 - 9\beta) = \tfrac{1}{4}lW(1 + 12\beta),$$

or

$$M_4 = \tfrac{1}{16}lW\frac{1 + 12\beta}{1 - 9\beta}.$$

From (ii),

$$R_4 = \tfrac{1}{2}W + 2\frac{M_4}{l} = \tfrac{1}{2}W\left\{1 + \tfrac{1}{4}\,\frac{1 + 12\beta}{1 - 9\beta}\right\},$$

or

$$R_4 = \tfrac{5}{8}W\frac{1 - 4.8\beta}{1 - 9\beta}.$$

Similarly, we find

$$R_3 = R_5 = \tfrac{3}{16}W\frac{1 - 16\beta}{1 - 9\beta}.$$

These agree, if $\beta = 0$, with the values of the reactions given in our Art. 2 as we should expect. If $\beta = \infty$, we obtain

$$R_3 = R_4 = R_5 = W/3,$$

the values which would be given by Dorna's theory for columns of equal cross-section.

Hence Clapeyron's or Dorna's theory will be more nearly in accordance with the actual facts of the case according as β is very small or very great. For intermediate values of β the theorem of the five moments must be used. Supposing the material of the column and girder to be the same, we have

$$\beta = \frac{k^2}{l^2}\,\frac{\omega}{\alpha}\,\frac{h}{l},$$

and it is difficult to imagine a practical case in which this expression could be *very* great. For columns very high and slender as compared with the span and cross-section of the girder it might become quite sensible, however, and in this case the methods of the present note would have to be adopted; certainly not Dorna's. Thus it would appear that Clapeyron's theorem would fail to give accurate results for a massive girder or tube supported on a great number of slender columns.

For cases of this kind equation (ix) will be found easy to work with when the numerical details are given. It may be advantageously used in cases where, as in suspending bars or bracing rods, a series of slender columns either as struts or ties support a massive continuous beam, for then the three moments theorem is certainly inapplicable.

(7) As a last example, to show the relation of Clapeyron's and Dorna's hypotheses, I will investigate the case of a girder uniformly loaded resting as above on three columns of the same material as the girder, but the terminal columns having cross-sections which are n' times that of the girder, while the central column has a cross-section n times that of the girder.

Returning to equation (v), we have, if W be the weight of the girder as before and $\gamma = k^2h/l^3$,

$$4M_4 - 12\gamma\left(\frac{1}{n'} + \frac{2}{n}\right)M_4 = \tfrac{1}{4}Wl\left\{1 + 12\left(\frac{2}{n} - \frac{1}{n'}\right)\gamma\right\},$$

or

$$M_4 = \tfrac{1}{16}Wl\frac{1 + 12\left(\frac{2}{n} - \frac{1}{n'}\right)\gamma}{1 - 3\left(\frac{2}{n} + \frac{1}{n'}\right)\gamma}.$$

Hence

$$R_2 = \tfrac{1}{2}W\left\{1 + \tfrac{1}{4}\frac{1 + 12\left(\frac{2}{n} - \frac{1}{n'}\right)\gamma}{1 - 3\left(\frac{2}{n} + \frac{1}{n'}\right)\gamma}\right\}$$

$$= \tfrac{5}{8}W\frac{1 - \frac{4.8}{n'}\gamma}{1 - 3\left(\frac{2}{n} + \frac{1}{n'}\right)\gamma}.$$

Similarly

$$R_3 = R_5 = \tfrac{3}{16}W\frac{1 - \frac{16}{n}\gamma}{1 - 3\left(\frac{2}{n} + \frac{1}{n'}\right)\gamma}.$$

These results agree with those obtained in the previous article, if $n = n'$.

Now let us see with what relation between n and n' it is possible for R_3 to equal R_4, whatever γ may be, *i.e.* even if it be small. We must have

$$10 - \frac{48}{n'}\gamma = 3 - \frac{48}{n}\gamma,$$

or

$$\frac{7}{48\gamma} = \frac{1}{n'} - \frac{1}{n}.$$

Thus n' must be less than n, or the middle column must be of greater cross-section than the terminal columns. Further, since γ is usually very small, n' must be *very much* less than n, or approximately

$$n' = \tfrac{48}{7}\gamma \left\{1 - \frac{48}{7n}\gamma\right\}.$$

Hence, by putting a very massive mid-column of otherwise arbitrary cross-section, and two very slender terminal columns with cross-sections about $\tfrac{48}{7}\gamma$ times that of the girder itself, we should have the three support-reactions nearly equal. Dorna's theory makes them equal when the three columns are of equal cross-section, which is obviously impossible for any reasonable value of γ.

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ON LATIN SQUARES.

By Prof. *Cayley*.

If in each line of a square of n^2 compartments the same n letters $a, b, c, \dots$ are arranged so that no letter occurs twice in the same column, we have what was termed by Euler "a Latin square." Supposing that in one of the lines the letters are arranged in the natural order *abcde*..., then in the remaining lines there must be arrangements beginning with b, c, d, e, &c., respectively, and we may consider the case in which the bottom line has the arrangement *abcde*..., and in the other lines, reckoning from the bottom one in order, the arrangements begin with b, c, d, e, &c., respectively: if the number of such squares be $= N$, then, obviously, the whole number of squares which can be formed with the same n arrangements is $= N[n]^n$.

Starting with the bottom line as above, then it is a well-known problem to determine the number of arrangements for the second line, viz., this number is

$$=1.2.3\ldots n\left\{1-\frac{1}{1}+\frac{1}{1.2}\cdots\pm\frac{1}{1.2\ldots n}\right\};$$

and if we assume, as above, that the second line begins with b, then the whole number of arrangements is this number divided by $(n-1)$, the quotient being of course integral. For instance, $n=5$, the number is $=120-120+60-20+5-1, =44$, which is divisible by 4, and the number of arrangements for the second line is thus $=11$.

But the number of arrangements for the third line will be different according to the arrangement selected for the second line, and it is not easy to see how in general the whole number of arrangements for the third line is to be calculated, and the difficulty of course increases for the next following lines; it may be remarked that, when all except the top-line are filled up, the top-line is completely determined.

Imagine the square completed, we may write down the substitutions by which we pass from the bottom line to itself (this is of course the substitution 1) and to each of the other lines respectively; we have thus a set of n substitutions, which may form *a group*; and when this is so, we may conversely from the group construct the latin square. But it is not every Latin square which is thus connected with a group of n substitutions.

In the cases $n=2, 3, 4$ there is no difficulty, the squares are

$$\begin{array}{llllll}
b\,a, & c\,a\,b, & d\,c\,a\,b, & d\,c\,b\,a, & d\,c\,b\,a, & d\,a\,b\,c, \\
a\,b & b\,c\,a & c\,d\,b\,a & c\,d\,a\,b & c\,a\,d\,b & c\,d\,a\,b \\
 & a\,b\,c & b\,a\,d\,c & b\,a\,d\,c & b\,d\,a\,c & b\,c\,d\,a \\
 & & a\,b\,c\,d & a\,b\,c\,d & a\,b\,c\,d & a\,b\,c\,d
\end{array}$$

viz., $n=2$ the number is 1, $n=3$ it is 1, $n=4$ it is 4; in this last case the arrangement *badc* for the second line gives two squares, but each of the other arrangements only one square.

In each of the squares of 4 we have a group, viz., for the four squares respectively, these are

1°,	1,	$(ab)(cd)$,	$(acbd)$,	$(adbc)$,
2	1	$(ab)(cd)$	$(ac)(bd)$	$(ad)(bc)$
3°	1	$(abdc)$	$(acdb)$	$(ad)(bc)$
4°	1	$(abcd)$	$(ac)(bd)$	$(adcb)$

1°, 3°, 4° are the cyclical groups of $(acbd)$, $(abdc)$, and $(abcd)$, respectively; 2° is a different kind of group.

In the case $n=5$, the whole number of squares is 56, viz., there are five arrangements of the second line each giving four squares, and six arrangements each giving six squares, $5.4+6.6=56$. The five arrangements are

$$\begin{array}{ccccc} b\,a\,e\,c\,d, & b\,a\,d\,e\,c, & b\,c\,a\,e\,d, & b\,d\,e\,a\,c, & b\,e\,d\,c\,a, \\ a\,b\,c\,d\,e & a\,b\,c\,d\,e & a\,b\,c\,d\,e & a\,b\,c\,d\,e & a\,b\,c\,d\,e \end{array}$$

viz., in these cases, the substitutions for passing to the second line are $(ab)(ced)$, $(ab)(cde)$, $(abc)(de)$, $(abd)(ce)$, $(abe)(cd)$, respectively.

The six arrangements are

$$\begin{array}{cccccc} b\,d\,a\,e\,c, & b\,e\,a\,c\,d, & b\,c\,e\,a\,d, & b\,e\,d\,a\,c, & b\,c\,d\,e\,a, & b\,d\,e\,c\,a, \\ a\,b\,c\,d\,e & a\,b\,c\,d\,e & a\,b\,c\,d\,e & a\,b\,c\,d\,e & a\,b\,c\,d\,e & a\,b\,c\,d\,e \end{array}$$

viz., in these cases, the substitutions for passing to the second line are $(abdec)$, $(abedc)$, $(abced)$, $(abecd)$, $(abcde)$, $(abdce)$, respectively.

A set of four squares is

$$\begin{array}{cccc} e\,c\,d\,b\,a, & e\,d\,a\,b\,c, & e\,c\,d\,a\,b, & e\,d\,b\,a\,c, \\ d\,e\,b\,a\,c & d\,c\,b\,e\,a & d\,e\,a\,b\,c & d\,c\,a\,e\,b \\ c\,d\,a\,e\,b & c\,e\,d\,a\,b & c\,d\,b\,e\,a & c\,e\,d\,b\,a \\ b\,a\,e\,c\,d & b\,a\,e\,c\,d & b\,a\,e\,c\,d & b\,a\,e\,c\,d \\ a\,b\,c\,d\,e & a\,b\,c\,d\,e & a\,b\,c\,d\,e & a\,b\,c\,d\,e \end{array}$$

and a set of six squares is

$$\begin{array}{cccccc} e\,c\,d\,a\,b, & e\,a\,d\,c\,b, & e\,c\,d\,b\,a, & e\,a\,b\,c\,d, & e\,a\,b\,c\,d, & e\,c\,b\,a\,d. \\ d\,e\,b\,c\,a & d\,c\,e\,b\,a & d\,a\,e\,c\,b & d\,c\,e\,b\,a & d\,c\,e\,a\,b & d\,a\,e\,c\,b \\ c\,a\,e\,b\,d & c\,e\,b\,a\,d & c\,e\,b\,a\,d & c\,e\,d\,a\,b & c\,e\,d\,b\,a & c\,e\,d\,b\,a \\ b\,d\,a\,e\,c & b\,d\,a\,e\,c & b\,d\,a\,e\,c & b\,d\,a\,e\,c & b\,d\,e\,a\,c & b\,d\,a\,e\,c \\ a\,b\,c\,d\,e & a\,b\,c\,d\,e & a\,b\,c\,d\,e & a\,b\,c\,d\,e & a\,b\,c\,d\,e & a\,b\,c\,d\,e \end{array}$$

In a square belonging to a set of four, the substitutions for obtaining from ~~any~~ one line all the other lines are of a form such as 1, $(ab)(ced)$, $(ac)(bde)$, $(ad)(bec)$, $(ae)(bcd)$, which are not a group. In the case of a set of six squares, there is one square of the set (in the foregoing instance the first square) where the substitutions are of a form such as 1, $(abdec)$, $(acedb)$, $(adceb)$, $(aebcd)$, and which thus form a cyclical group of five substitutions.

NOTE ON SERIES WHOSE COEFFICIENTS INVOLVE POWERS OF THE BERNOULLIAN NUMBERS.

By *J. W. L. Glaisher.*

§ 1. Denoting the Bernoullian numbers by B_1, B_2, B_3, ... we know that

$$\log\frac{\sin x}{x} = -\frac{B_1}{2.2!}2^2x^2 - \frac{B_2}{4.4!}2^4x^4 - \frac{B_3}{6.6!}2^6x^6 - \&c.$$

By writing in this equation $\frac{1}{2}x$, $\frac{1}{3}x$, $\frac{1}{4}x$, ... for x, and adding, the right-hand member becomes

$$\begin{aligned}
&-\frac{B_1}{2.2!}2^2x^2\left\{1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\&c.\right\}\\
&-\frac{B_2}{4.4!}2^4x^4\left\{1+\frac{1}{2^4}+\frac{1}{3^4}+\frac{1}{4^4}+\&c.\right\}\\
&-\frac{B_3}{6.6!}2^6x^6\left\{1+\frac{1}{2^6}+\frac{1}{3^6}+\frac{1}{4^6}+\&c.\right\}\\
&\qquad\qquad\&c, \qquad\qquad\qquad \&c.,
\end{aligned}$$

and, since

$$1+\frac{1}{2^{2n}}+\frac{1}{3^{2n}}+\frac{1}{4^{2n}}+\&c. = \frac{(2\pi)^{2n}B_n}{2.(2n)!},$$

the above series

$$= -\frac{B_1^2}{2.2.(2!)^2}(4\pi x)^2 - \frac{B_2^2}{2.4.(4!)^2}(4\pi x)^4 - \frac{B_3^2}{2.6.(6!)^2}(4\pi x)^6 - \&c.$$

Now the left-hand member

$$= \log\left\{\frac{\sin x}{x}\cdot\frac{\sin\frac{1}{2}x}{\frac{1}{2}x}\cdot\frac{\sin\frac{1}{3}x}{\frac{1}{3}x}\cdots\right\},$$

and, by Euler's formula, this product

$$\begin{aligned}
= \;&\cos\frac{x}{2}\cos\frac{x}{4}\cos\frac{x}{8}\cos\frac{x}{16}\cdots\\
&\times\cos\frac{x}{4}\cos\frac{x}{8}\cos\frac{x}{16}\cos\frac{x}{32}\cdots\\
&\times\cos\frac{x}{6}\cos\frac{x}{12}\cos\frac{x}{24}\cos\frac{x}{48}\cdots\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
= \;&\cos\frac{x}{2}\left(\cos\frac{x}{4}\right)^2\cos\frac{x}{6}\left(\cos\frac{x}{8}\right)^3\cos\frac{x}{10}\left(\cos\frac{x}{12}\right)^2\cos\frac{x}{14}\cdots,
\end{aligned}$$

the exponent of each cosine being equal to that of the highest power of 2 contained in the denominator of the argument.

We thus find

$$\log\left\{\cos\frac{x}{2}\left(\cos\frac{x}{4}\right)^2\cos\frac{x}{6}\left(\cos\frac{x}{8}\right)^3\dots\right\}=-\Sigma_1^\infty\frac{B_n^{\ 2}}{2.2n.\{(2n)!\}^2}(4\pi x)^{2n},$$

whence, by differentiating,

$$\tfrac{1}{2}\tan\frac{x}{2}+\tfrac{2}{4}\tan\frac{x}{4}+\tfrac{1}{6}\tan\frac{x}{6}+\tfrac{3}{8}\tan\frac{x}{8}+\&\text{c.}$$

$$=2\pi\left\{\frac{B_1^{\ 2}}{(2\,!)^2}4\pi x+\frac{B_2^{\ 2}}{(4\,!)^2}(4\pi x)^3+\frac{B_3^{\ 2}}{(6\,!)^2}(4\pi x)^5+\&\text{c.}\right\}.$$

If we denote by r the exponent of the highest power of 2 which is a divisor of $2n$ (*i.e.* so that $2n=2^rm$, m being an uneven number), we may write this result in the form

$$\Sigma_{n=1}^{n=\infty}\ \frac{r}{n}\tan\frac{x}{2n}=4\pi\Sigma_{n=1}^{n=\infty}\ \frac{B_n^{\ 2}}{\{(2n)\,!\}^2}(4\pi x)^{2n-1},$$

or, multiplying by x and transposing the sides of the equation,

$$\Sigma_{n=1}^{n=\infty}\left\{\frac{B_n}{(2n)\,!}\right\}^2(4\pi x)^{2n}=2x\Sigma_{n=1}^{n=\infty}\ \frac{r}{2n}\tan\frac{x}{2n}\,.\text{*}$$

§ 2. In the preceding formula, put for x successively $\frac{1}{2}x$, $\frac{1}{3}x$, $\frac{1}{4}x$, ... The general term on the left-hand side then becomes

$$\left\{\frac{B_n}{(2n)\,!}\right\}^2(4\pi x)^{2n}\left\{1+\frac{1}{2^{2n}}+\frac{1}{3^{2n}}+\frac{1}{4^{2n}}+\&\text{c.}\right\}$$

$$=\tfrac{1}{2}\left\{\frac{B_n}{(2n)\,!}\right\}^3(8\pi^2x)^{2n}.$$

To obtain the value of the right-hand member of the equation, consider the expression

$$\phi\left(\frac{x}{2}\right)+2\phi\left(\frac{x}{4}\right)+\phi\left(\frac{x}{6}\right)+3\phi\left(\frac{x}{8}\right)+\phi\left(\frac{x}{10}\right)+2\phi\left(\frac{x}{12}\right)$$

$$+\phi\left(\frac{x}{14}\right)+4\phi\left(\frac{x}{16}\right)+2\phi\left(\frac{x}{18}\right)+6\phi\left(\frac{x}{20}\right)+\&\text{c.}$$

in which the coefficient of each term is equal to the exponent of the highest power of 2 contained in the denominator.

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* I set this formula in Part II of the Mathematical Tripos, 1887 (Friday morning, June 3, Question 7).

By putting $\frac{1}{2}x$, $\frac{1}{3}x$, $\frac{1}{4}x$, ... for x, in this expression, and adding, we obtain the expression

$$\phi\left(\frac{x}{2}\right)+3\phi\left(\frac{x}{4}\right)+2\phi\left(\frac{x}{6}\right)+6\phi\left(\frac{x}{8}\right)+2\phi\left(\frac{x}{10}\right)+6\phi\left(\frac{x}{12}\right)$$
$$+2\phi\left(\frac{x}{14}\right)+10\phi\left(\frac{x}{16}\right)+3\phi\left(\frac{x}{18}\right)+6\phi\left(\frac{x}{20}\right)+\text{\&c.},$$

The law of these coefficients, which is a rather curious one, may be stated as follows:—the coefficient of $\phi\left(\frac{x}{2n}\right)$ is equal to $\frac{1}{2}r(r+1)\,\delta_1(2n)$, where, as before, r is the exponent of the highest power of 2 contained in $2n$, and $\delta_1(2n)$ denotes the number of uneven divisors of $2n$.

If, therefore,

$$2n=2^r a^\alpha b^\beta c^\gamma \ldots,$$

where a, b, c, ... are uneven primes, then

$$\delta_1(2n)=(\alpha+1)(\beta+1)(\gamma+1)\ldots,$$

and the coefficient of $\phi\left(\frac{x}{2n}\right)$

$$=\frac{r(r+1)}{2}(\alpha+1)(\beta+1)(\gamma+1)\ldots.$$

The quantity $\frac{r(r+1)}{2}$ is the rth triangular number. Thus the coefficient of $\phi\left(\frac{x}{2n}\right)$ is equal to the product of the rth triangular number and the number of uneven divisors of $2n$.

If we denote this coefficient by $\lambda_1(2n)$, we have

$$\Sigma_{n=1}^{n=\infty}\left\{\frac{B_n}{(2n)!}\right\}^3(8\pi^2x)^{2n}=4x\Sigma_{n=1}^{n=\infty}\frac{\lambda_1(2n)}{2n}\tan\frac{x}{2n}.$$

§ 3. Proceeding as before (*i.e.* substituting $\frac{1}{2}x$, $\frac{1}{3}x$, ... for x, and adding), we find that the left-hand member of the equation becomes

$$\tfrac{1}{2}\Sigma_1^\infty\left\{\frac{B_n}{(2n)!}\right\}^4(16\pi^3x)^{2n}.$$

To obtain the value of the right-hand member, we consider the expression

$$\phi\left(\frac{x}{2}\right)+3\phi\left(\frac{x}{4}\right)+2\phi\left(\frac{x}{6}\right)+6\phi\left(\frac{x}{8}\right)+2\phi\left(\frac{x}{10}\right)+\text{\&c.},$$

and substitute $\frac{1}{2}x$, $\frac{1}{3}x$, ... for x.

The expression obtained therefrom by addition is

$$\phi\left(\frac{x}{2}\right)+4\phi\left(\frac{x}{4}\right)+3\phi\left(\frac{x}{6}\right)+10\phi\left(\frac{x}{8}\right)+3\phi\left(\frac{x}{10}\right)+12\phi\left(\frac{x}{12}\right)$$
$$+3\phi\left(\frac{x}{14}\right)+20\phi\left(\frac{x}{16}\right)+6\phi\left(\frac{x}{18}\right)+12\phi\left(\frac{x}{20}\right)+\&c.$$

The coefficient of $\phi\left(\frac{x}{2n}\right)$ in this series is

$$\frac{r(r+1)(r+2)}{6}\,\delta_2(2n),$$

where r has the same meaning as before, and $\delta_2(2n)$ denote the sum of the divisors of each of the uneven divisors of $2n$.

Thus, if $1, p, q, \ldots, m$ are all the uneven divisors of $2n$, then

$$\delta_2(2n)=\delta_1(1)+\delta_1(p)+\delta_1(q)+\ldots+\delta_1(m).$$

If $$2n=2^r a^\alpha b^\beta c^\gamma\ldots,$$

$a, b, c, \ldots$ being uneven primes, it is easy to see that

$$\delta_2(2n)=\frac{(\alpha+1)(\alpha+2)}{2}\cdot\frac{(\beta+1)(\beta+2)}{2}\cdot\frac{(\gamma+1)(\gamma+2)}{2}\ldots;$$

and therefore the coefficient of $\phi\left(\frac{x}{2n}\right)$ is

$$\frac{r(r+1)(r+2)}{6}\cdot\frac{(\alpha+1)(\alpha+2)}{2}\cdot\frac{(\beta+1)(\beta+2)}{2}\ldots:$$

Denoting this coefficient by $\lambda_2(2n)$, we have

$$\Sigma_{n=1}^{n=\infty}\left\{\frac{B_n}{(2n)!}\right\}^4(16\pi^3x)^{2n}=8x\Sigma_{n=1}^{n=\infty}\frac{\lambda_2(2n)}{2n}\tan\frac{x}{2n}.$$

§ 4. It is evident that, if p and q be any two numbers which are prime to each other,

$$\delta_1(p)\,\delta_1(q)=\delta_1(pq),$$

and $$\delta_2(p)\,\delta_2(q)=\delta_2(pq).$$

These formulæ would greatly facilitate the actual calculation of the coefficients.

If n is an uneven prime number,

$$\delta_1(n)=2,\quad \delta_2(n)=3,$$

and $$\lambda_1(2n)=2,\ \lambda_2(2n)=3.$$

If $2n = 2^r a$, a being an uneven prime number,

$$\lambda_1(2n) = \frac{r(r+1)}{2}, \quad \lambda_2(2n) = \frac{r(r+1)(r+2)}{6}.$$

§ 5. By writing $2x$ for x, and replacing $\lambda_1(2n)$ and $\lambda_2(2n)$ by other functions $\theta_1(n)$ and $\theta_2(n)$ defined below, we may write the system of formulæ in the following form:

$$\Sigma_1^\infty \left\{\frac{B_n}{(2n)!}\right\}^2 (8\pi x)^{2n} = 2x\Sigma_1^\infty \frac{\theta_0(n)}{n} \tan\frac{x}{n},$$

$$\Sigma_1^\infty \left\{\frac{B_n}{(2n)!}\right\}^3 (16\pi^2 x)^{2n} = 4x\Sigma_1^\infty \frac{\theta_1(n)}{n} \tan\frac{x}{n},$$

$$\Sigma_1^\infty \left\{\frac{B_n}{(2n)!}\right\}^4 (32\pi^3 x)^{2n} = 8x\Sigma_1^\infty \frac{\theta_2(n)}{n} \tan\frac{x}{n};$$

where, if $$n = 2^s a^\alpha b^\beta c^\gamma \ldots,$$

$a, b, c, \ldots$ being uneven primes, then

$$\theta_0(n) = s + 1,$$

$$\theta_1(n) = \frac{(s+1)(s+2)}{2}\delta_1(n)$$

$$= \frac{(s+1)(s+2)}{2}(\alpha+1)(\beta+1)(\gamma+1)\ldots,$$

$$\theta_2(n) = \frac{(s+1)(s+2)(s+3)}{6}\delta_2(n)$$

$$= \frac{(s+1)(s+2)(s+3)}{6} \cdot \frac{(\alpha+1)(\alpha+2)}{2} \cdot \frac{(\beta+1)(\beta+2)}{2} \ldots.$$

If p and q be relatively prime, we have

$$\theta_0(p)\,\theta_0(q) = \theta_0(pq),$$

$$\theta_1(p)\,\theta_1(q) = \theta_1(pq),$$

$$\theta_2(p)\,\theta_2(q) = \theta_2(pq).$$

The general law of the series is evident: the value of $\theta_r(n)$ being

$$\frac{(s+1)^{(r+1)}}{(r+1)!} \cdot \frac{(\alpha+1)^{(r)}}{r!} \cdot \frac{(\beta+1)^{(r)}}{r!} \ldots,$$

where $a^{(b)}$ denotes the factorial $a(a+1)\ldots(a+b-1)$.

§ 6. The formula, corresponding to those in the last section, in which the first powers only of the Bernoullian numbers are involved, may be written:

$$\Sigma_1^\infty \frac{B_n}{(2n)!}(2x)^{2n} = x\Sigma_1^\infty \frac{1}{2^n}\tan\frac{x}{2^n}$$
$$= 1 - x\cot x.$$

The two expressions on the right-hand side are at once seen to be equal by differentiating logarithmically Euler's formula

$$\frac{\sin x}{x} = \cos\frac{x}{2}\cos\frac{x}{4}\cos\frac{x}{8}\dots.$$

It is perhaps worth noticing how readily the above expansion of $1 - x\cot x$ in powers of x is derivable from the expression for the Bernoullian numbers in terms of the sums of the reciprocals of the even powers of the natural numbers. For

$$1 - x\cot x = -\Sigma_1^\infty \frac{2x^2}{x^2 - n^2\pi^2}$$
$$= \Sigma_1^\infty \frac{2x^{2n}}{\pi^{2n}}\left(1 + \frac{1}{2^{2n}} + \frac{1}{3^{2n}} + \frac{1}{4^{2n}} + \dots\right)$$
$$= \Sigma_1^\infty \frac{2x^{2n}}{\pi^{2n}} \cdot \frac{(2\pi)^{2n}B_n}{2(2n)!} = \Sigma_1^\infty \frac{B_n}{2n!}(2x)^{2n}.$$

§ 7. We may deduce by integration from § 5, or obtain independently as in § 1, the following formulæ:

$$\Sigma_1^\infty \frac{1}{2n}\left\{\frac{B_n}{(2n)!}\right\}^2 (8\pi x)^{2n} = -2\log\Pi_1^\infty\left(\cos\frac{x}{n}\right)^{\theta_0(n)},$$
$$\Sigma_1^\infty \frac{1}{2n}\left\{\frac{B_n}{(2n)!}\right\}^3 (16\pi^2 x)^{2n} = -4\log\Pi_1^\infty\left(\cos\frac{x}{n}\right)^{\theta_1(n)},$$
$$\Sigma_1^\infty \frac{1}{2n}\left\{\frac{B_n}{(2n)!}\right\}^4 (32\pi^3 x)^{2n} = -8\log\Pi_1^\infty\left(\cos\frac{x}{n}\right)^{\theta_2(n)},$$
$$\&c. \qquad \&c.$$

§ 8. By differentiating the formulæ in § 5, we find

$$\Sigma_2^\infty (2n-1)\left\{\frac{B_n}{(2n)!}\right\}^2 (8\pi x)^{2n} = 2x^2\Sigma_1^\infty \frac{\theta_0(n)}{n^2}\tan^2\frac{x}{n},$$
$$\Sigma_2^\infty (2n-1)\left\{\frac{B_n}{(2n)!}\right\}^3 (16\pi^2 x)^{2n} = 4x^2\Sigma_1^\infty \frac{\theta_1(n)}{n^2}\tan^2\frac{x}{n},$$
$$\Sigma_2^\infty (2n-1)\left\{\frac{B_n}{(2n)!}\right\}^4 (32\pi^3 x)^{2n} = 8x^2\Sigma_1^\infty \frac{\theta_2(n)}{n^2}\tan^2\frac{x}{n};$$
$$\&c. \qquad \&c.,$$

together with

$$\Sigma_1^\infty \frac{\theta_0(n)}{n^2} = 8B_1^2\pi^2,$$

$$\Sigma_1^\infty \frac{\theta_1(n)}{n^2} = 8B_1^3\pi^4,$$

$$\Sigma_1^\infty \frac{\theta_2(n)}{n^2} = 8B_1^4\pi^4,$$

&c. &c.

§ 9. These latter formulæ are easily verified; for $\Sigma_1^\infty \frac{\theta_0(n)}{n^2}$ may be derived from

$$1 + \frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{8^2} + \frac{1}{16^2} + \&c.,$$

by dividing it by $\frac{1}{2^2}, \frac{1}{3^2}, \frac{1}{4^2}, \ldots$, and adding; whence

$$\Sigma_1^\infty \frac{\theta_0(n)}{n^2} = \left(1 + \frac{1}{2^2} + \frac{1}{4^2} + \&c.\right)\left(1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \&c.\right)$$

$$= \left(1 - \frac{1}{2^2}\right)^{-1}\left(1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \&c.\right)$$

$$= \frac{4}{3} \cdot \frac{\pi^2}{6} = \frac{2\pi^2}{9}.$$

Similarly

$$\Sigma_1^\infty \frac{\theta_1(n)}{n^2} = \left(1 + \frac{1}{2^2} + \frac{1}{4^2} + \&c.\right)\left(1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \&c.\right)^2$$

$$= \frac{2\pi^2}{9} \times \frac{\pi^2}{6} = \frac{\pi^4}{27},$$

and

$$\Sigma_1^\infty \frac{\theta_2(n)}{n^2} = \frac{\pi^4}{27} \times \frac{\pi^2}{6} = \frac{\pi^6}{162}.$$

§ 10. It may be remarked that, in general, if

$$\phi(x) = \alpha_1 x^2 + \alpha_2 x^4 + \alpha_3 x^6 + \&c.,$$

then, by substituting $\frac{1}{2}x$, $\frac{1}{3}x$, ..., for x, and adding, we find

$$\Sigma_1^\infty \alpha_n \frac{B_n}{(2n)!}(2\pi x)^{2n} = 2\Sigma_1^\infty \phi\left(\frac{x}{n}\right).$$

Similarly we find that

$$\Sigma_1^\infty \alpha_n \left\{\frac{B_n}{(2n)\,!}\right\}^2 (4\pi^2 x)^{2n} = 4\Sigma_1^\infty \nu_1(n)\,\phi\left(\frac{x}{n}\right),$$

where $\nu_1(n)$ denotes the number of divisors of n.

Thus, if $$n = a^\alpha b^\beta c^\gamma \ldots,$$

$a, b, c, \ldots$ being any primes, then

$$\nu_1(n) = (\alpha+1)(\beta+1)(\gamma+1)\ldots .$$

We also find that

$$\Sigma_1^\infty \alpha_n \left\{\frac{B_n}{(2n)\,!}\right\}^3 (8\pi^2 x)^{2n} = 8\Sigma_1^\infty \nu_2(n)\,\phi\left(\frac{x}{n}\right),$$

where $\nu_2(n)$ denotes the number of the divisors of all the divisors of n. Thus, if $1, p, q, \ldots, n$ be the divisors of n,

$$\nu_2(n) = \nu_1(1) + \nu_1(p) + \nu_1(q) + \ldots + \nu_1(n);$$

and, if $$n = a^\alpha b^\beta c^\gamma \ldots,$$

$a, b, c, \ldots$ being any primes,

$$\nu_2(n) = \frac{(\alpha+1)(\alpha+2)}{2} \cdot \frac{(\beta+1)(\beta+2)}{2} \cdot \frac{(\gamma+1)(\gamma+2)}{2} \ldots .$$

The formula involving B_n^4 is

$$\Sigma_1^\infty \alpha_n \left\{\frac{B_n}{(2n)\,!}\right\}^4 (16\pi^3 x)^{2n} = 16\Sigma_1^\infty \nu_3(n)\,\phi\left(\frac{x}{n}\right),$$

where, if $$n = a^\alpha b^\beta c^\gamma \ldots,$$

$$\nu_3(n) = \frac{(\alpha+1)(\alpha+2)(\alpha+3)}{6} \cdot \frac{(\beta+1)(\beta+2)(\beta+3)}{6} \ldots;$$

and similarly we find

$$\Sigma_1^\infty \alpha_n \left\{\frac{B_n}{(2n)\,!}\right\}^5 (32\pi^4 x)^{2n} = 32\Sigma_1^\infty \nu_4(n)\,\phi\left(\frac{x}{n}\right),$$

where

$$\nu_4(n) = \frac{(\alpha+1)(\alpha+2)\ldots(\alpha+4)}{4\,!} \cdot \frac{(\beta+1)(\beta+2)\ldots(\beta+4)}{4\,!} \ldots,$$

the general law being evident.

Thus $\nu_1, \nu_2, \nu_3, \ldots$ all satisfy the equation

$$\nu(p)\,\nu(q) = \nu(pq),$$

p and q being prime to each other.

§ 11. In the case of the formulæ considered in §§ 1–8, we have

$$\alpha_n = \frac{B_n}{(2n)!}\,2^{2n},$$

and

$$\phi(x) = x\Sigma_1^\infty \frac{1}{2^n}\tan\frac{x}{2^n}.$$

The fact that $\phi(x)$ is itself a series of terms in which the denominators are the successive powers of 2 is the cause of the distinction which occurs in the formulæ of § 5 between 2 and the other prime factors of n. In the formation of the functions θ the exponent of 2 gives rise to a factorial which is one order higher than the factorials depending upon the exponents of the other primes.

EXPANSIONS OF K, I, G, E IN POWERS OF $k'^2 - k^2$.

By *J. W. L. Glaisher.*

THE object of this note is to give the expansion of K in ascending powers of $k'^2 - k^2$. I have also added the corresponding expansions of I, G, and E.

Expansion of K, §§ 1, 2.

§ 1. Let h and h' denote k^2 and k'^2 respectively, and let

$$\lambda = h' - h = k'^2 - k^2.$$

If therefore α be the modular angle, so that $k = \sin\alpha$, then $\lambda = \cos 2\alpha$.

We have*

$$\frac{\sqrt{\pi}}{4}K = \int_0^\infty\int_0^\infty e^{-x^4-y^4-2\lambda x^2y^2}dx\,dy,$$

whence, expanding in powers of λ,

$$\frac{\sqrt{\pi}}{4}K = \int_0^\infty\int_0^\infty e^{-x^4-y^4}\left\{1 - 2\lambda x^2y^2 + \frac{2^2\lambda^2}{2!}x^4y^4 - \frac{2^3\lambda^3}{3!}x^6y^6 + \&c.\right\}.$$

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* *Proceedings of the London Mathematical Society,* vol. XIII. p. 92 (1881).

Now
$$\int_0^\infty e^{-x^4}\,dx = \tfrac{1}{4}\Gamma\left(\tfrac{1}{4}\right),$$
$$\int_0^\infty e^{-x^4}x^2\,dx = \tfrac{1}{4}\Gamma\left(\tfrac{3}{4}\right),$$
$$\int_0^\infty e^{-x^4}x^4\,dx = \tfrac{1}{4}\Gamma\left(\tfrac{5}{4}\right),$$
$$\&c. \qquad \&c.;$$

whence

$$\frac{\sqrt{\pi}}{4}K = \frac{1}{4^2}\Gamma^2\left(\tfrac{1}{4}\right)\left\{1 + \frac{1}{2.4}\lambda^2 + \frac{1^2.5^2}{2.4.6.8}\lambda^4 + \frac{1^2.5^2.9^2}{2.4.6.8.10.12}\lambda^6 + \&c.\right\}$$
$$- \tfrac{1}{4}\Gamma^2\left(\tfrac{3}{4}\right)\left\{\frac{1}{2}\lambda + \frac{3^2}{2.4.6}\lambda^3 + \frac{3^2.7^2}{2.4.6.8.10}\lambda^5 + \frac{3^2.7^2.11^2}{2.4.6\ldots14} + \&c.\right\}.$$

Denoting by K° the value of K corresponding to the modulus $\frac{1}{\sqrt{2}}$, we know that

$$4\sqrt{\pi}.K^\circ = \Gamma^2\left(\tfrac{1}{4}\right).$$

Also, since $\qquad \Gamma\left(\tfrac{1}{4}\right)\Gamma\left(\tfrac{3}{4}\right) = \sqrt{2}.\pi,$

it follows that $\qquad \dfrac{\pi^{\frac{3}{2}}}{2K^\circ} = \Gamma^2\left(\tfrac{3}{4}\right).$

Thus we find

$$K = K^\circ\left\{1 + \frac{1^2}{2.4}\lambda^2 + \frac{1^2.5^2}{2.4.6.8}\lambda^4 + \frac{1^2.5^2.9^2}{2.4.6.8.10.12}\lambda^6 + \&c.\right\}$$
$$- \frac{\pi}{2K^\circ}\left\{\frac{1}{2}\lambda + \frac{3^2}{2.4.6}\lambda^3 + \frac{3^2.7^2}{2.4.6.8.10}\lambda^5 + \frac{3^2.7^2.11^2}{2.4.6\ldots14}\lambda^7 + \&c.\right\}.$$

Since $\qquad 2G^\circ K^\circ = \tfrac{1}{2}\pi,$

we have $\qquad 2G^\circ = \dfrac{\pi}{2K^\circ};$

and we may therefore write the expansion of K in the form

$$K = K^\circ\left\{1 + \frac{1^2}{2.4}\lambda^2 + \frac{1^2.5^2}{2.4.6.8}\lambda^4 + \frac{1^2.5^2.9^2}{2.4.6.8.10.12}\lambda^6 + \&c.\right\}$$
$$- 2G^\circ\left\{\frac{1}{2}\lambda + \frac{3^2}{2.4.6}\lambda^3 + \frac{3^2.7^2}{2.4.6.8.10}\lambda^5 + \frac{3^2.7^2.11^2}{2.4.6\ldots14}\lambda^7 + \&c.\right\}.$$

The quantities K° and G° are of course mere numerical constants, their values to eight places of decimals being

$$K^\circ = 1{\cdot}85407468,$$

$$G^\circ = 1{\cdot}17875274.$$

§ 2. By changing the modulus from k to k' we change the sign of λ; we thus find

$$K' + K = 2K^\circ \left\{1 + \frac{1^2}{2.4}\lambda^2 + \frac{1^2.5^2}{2.4.6.8}\lambda^4 + \frac{1^2.5^2.9^2}{2.4.6.8.10.12}\lambda^6 + \&c.\right\}$$

$$K' - K = 4G^\circ \left\{\frac{1}{2}\lambda + \frac{3^2}{2.4.6}\lambda^3 + \frac{3^2.7^2}{2.4.6.8.10}\lambda^5 + \frac{3^2.7^2.11^2}{2.4.6\ldots14}\lambda^7 + \&c.\right\}.$$

Expansion of G, § 3.

§ 3. We may deduce the expansion of G from that of K by means of the formula*

$$G = 2hh'\frac{dK}{dh};$$

for $\qquad h = \tfrac{1}{2}(1-\lambda), \quad h' = \tfrac{1}{2}(1+\lambda),$

whence $\qquad G = -(1-\lambda^2)\dfrac{dK}{d\lambda}.$

We thus find

$$G = \quad G^\circ \left\{1 + \frac{1}{2.4}\lambda^2 + \frac{3^2}{2.4.6.8}\lambda^4 + \frac{3^2.7^2}{2.4.6.8.10.12}\lambda^6 + \&c.\right\}$$
$$- \frac{K^\circ}{2}\left\{\frac{1}{2}\lambda + \frac{1^2}{2.4.6}\lambda^3 + \frac{1^2.5^2}{2.4.6.8.10}\lambda^5 + \frac{1^2.5^2.9^2}{2.4.6\ldots14}\lambda^7 + \&c.\right\}.$$

By changing the modulus from k to k' we deduce also

$$G' + G = 2G^\circ \left\{1 + \frac{1}{2.4}\lambda^2 + \frac{3^2}{2.4.6.8}\lambda^4 + \frac{3^2.7^2}{2.4.6.8.10.12}\lambda^6 + \&c.\right\}$$

$$G' - G = K^\circ \left\{\frac{1}{2}\lambda + \frac{1^2}{2.4.6}\lambda^3 + \frac{1^2.5^2}{2.4.6.8.10}\lambda^5 + \frac{1^2.5^2.9^2}{2.4.6\ldots41}\lambda^7 + \&c.\right\}.$$

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* *Proceedings of the Cambridge Philosophical Society*, vol. v. p. 198 (1884).

Expansions of I and E, §§ 4–6.

§ 4. Since

$$hK = \tfrac{1}{2}(1-\lambda)K, \quad h'K = \tfrac{1}{2}(1+\lambda)K,$$

we may deduce from the expansions of K and G the following expansions of I and E:

$$\begin{aligned} I = \quad & G^\circ \left\{1 + \frac{1}{2}\lambda - \frac{3}{2.4}\lambda^2 + \frac{3^2}{2.4.6}\lambda^3 - \frac{3^2.7}{2.4.6.8}\lambda^4 + \frac{3^2.7^2}{2.4.6.8.10}\lambda^5 \right. \\ & \qquad \left. - \frac{3^2.7^2.11}{2.4.6.8.10.12}\lambda^6 + \&c.\right\} \\ & - \frac{K^\circ}{2}\left\{1 - \frac{1}{2}\lambda + \frac{1^2}{2.4}\lambda^2 - \frac{1^2.5}{2.4.6}\lambda^3 + \frac{1^2.5^2}{2.4.6.8}\lambda^4 - \frac{1^2.5^2.9}{2.4.6.8.10}\lambda^5 \right. \\ & \qquad \left. + \frac{1^2.5^2.9^2}{2.4.6.8.10.12}\lambda^6 - \&c.\right\}; \end{aligned}$$

$$\begin{aligned} E = \quad & G^\circ \left\{1 - \frac{1}{2}\lambda - \frac{3}{2.4}\lambda^2 - \frac{3^2}{2.4.6}\lambda^3 - \frac{3^2.7}{2.4.6.8}\lambda^4 - \frac{3^2.7^2}{2.4.6.8.10}\lambda^5 \right. \\ & \qquad \left. - \frac{3^2.7^2.11}{2.4.6.8.10.12}\lambda^6 - \&c.\right\} \\ & + \frac{K^\circ}{2}\left\{1 + \frac{1}{2}\lambda + \frac{1^2}{2.4}\lambda^2 + \frac{1^2.5}{2.4.6}\lambda^3 + \frac{1^2.5^2}{2.4.6.8}\lambda^4 + \frac{1^2.5^2.9}{2.4.6.8.10}\lambda^5 \right. \\ & \qquad \left. + \frac{1^2.5^2.9^2}{2.4.6.8.10.12}\lambda^6 + \&c.\right\}. \end{aligned}$$

If we put $k = k'$, then $\lambda = 0$, and these formulæ become

$$I^\circ = G^\circ - \tfrac{1}{2}K^\circ, \quad E^\circ = G^\circ + \tfrac{1}{2}K^\circ,$$

which are the expressions for I° and E° in terms of K° and G°.

§ 5. From these expansions we may deduce the following:

$$\begin{aligned} E + I' = 2G^\circ & \left\{1 - \frac{1}{2}\lambda - \frac{3}{2.4}\lambda^2 - \frac{3^2}{2.4.6}\lambda^3 - \frac{3^2.7}{2.4.6.8}\lambda^4 \right. \\ & \qquad \left. - \frac{3^2.7^2}{2.4.6.8.10}\lambda^5 - \frac{3^2.7^2.11}{2.4.6.8.10.12}\lambda^6 - \&c.\right\}, \end{aligned}$$

$$\begin{aligned} E - I' = K^\circ & \left\{1 + \frac{1}{2}\lambda + \frac{1^2}{2.4}\lambda^2 + \frac{1^2.5}{2.4.6}\lambda^3 + \frac{1^2.5^2}{2.4.6.8}\lambda^4 \right. \\ & \qquad \left. + \frac{1^2.5^2.9}{2.4.6.8.10}\lambda^5 + \frac{1^2.5^2.9^2}{2.4.6.8.10.12}\lambda^6 + \&c.\right\}, \end{aligned}$$

with the corresponding expansions for $E' + I$ and $E' - I$ obtained by changing the sign of λ.

§ 6. It may be remarked that we have also

$$I+E=\ 2G^\circ\left\{1-\frac{3}{2.4}\lambda^2-\frac{3^2.7}{2.4.6.8}\lambda^4-\frac{3^2.7^2.11}{2.4.6.8.10.12}\lambda^6-\&\text{c.}\right\}$$

$$+K^\circ\left\{\frac{1}{2}\lambda+\frac{1^2.5}{2.4.6}\lambda^3+\frac{1^2.5^2.9}{2.4.6.8.10}\lambda^5+\&\text{c.}\right\},$$

$$I-E=\ 2G^\circ\left\{\frac{1}{2}\lambda+\frac{3^2}{2.4.6}\lambda^3+\frac{3^2.7^2}{2.4.6.8.10}\lambda^5+\&\text{c.}\right\}$$

$$-K^\circ\left\{1+\frac{1^2}{2.4}\lambda^2+\frac{1^2.5^2}{2.4.6.8}\lambda^4+\frac{1^2.5^2.9^2}{2.4.6.8.10.12}\lambda^6+\&\text{c.}\right\},$$

the latter series being the same as that for $-K$ (§ 1), as it ought to be, since $I-E=-K$.

From these two formulæ we may deduce

$$I+E+I'+E'=\ 4G^\circ\left\{1-\frac{3}{2.4}\lambda^2-\frac{3^2.7}{2.4.6.8}\lambda^4-\&\text{c.}\right\},$$

$$I+E-I'-E'=\ 2K^\circ\left\{\frac{1}{2}\lambda+\frac{1^2.5}{2.4.6}\lambda^3+\frac{1^2.5^2.9}{2.4\ldots10}\lambda^5+\&\text{c.}\right\},$$

$$I-E+I'-E'=-2K^\circ\left\{1+\frac{1^2}{2.4}\lambda^2+\frac{1^2.5^2}{2.4.6.8}\lambda^4+\&\text{c.}\right\},$$

$$I-E-I'+E'=\ 4G^\circ\left\{\frac{1}{2}\lambda+\frac{3^2}{2.4.6}\lambda^3+\frac{3^2.7^2}{2.4\ldots10}\lambda^5+\&\text{c.}\right\}.$$

The values of the last two series are $-K-K'$ and $-K+K'$, agreeing with § 2. In previous papers I have denoted $\frac{1}{2}(I+E)$ by W (W being a quantity bearing a close analogy to G). The first, third, and fourth series represent therefore W, $W+W'$, and $W-W'$ respectively.

Expansion of $I+G+E$, § 7.

§ 7. For $I+G+E$ we obtain the expansion:

$$I+G+E=\ G^\circ\left\{3-\frac{5}{2.4}\lambda^2-\frac{3^2.13}{2.4.6.8}\lambda^4-\frac{3^2.7^2.21}{2.4.6.8.10.12}\lambda^6\right.$$

$$\left.-\frac{3^2.7^2.11^2.29}{2.4.6\ldots16}\lambda^8-\&\text{c.}\right\}$$

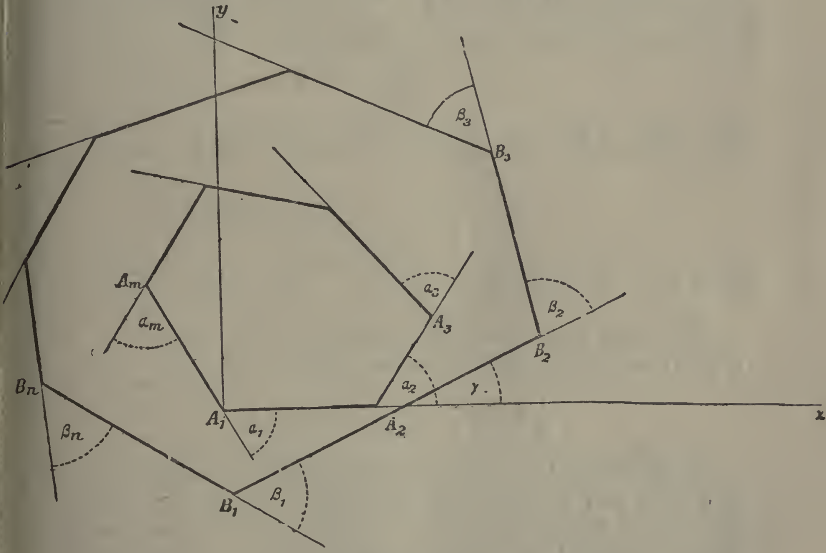
$$+\frac{K^\circ}{2}\left\{\frac{1}{2}\lambda+\frac{1^2.9}{2.4.6}\lambda^3+\frac{1^2.5^2.17}{2.4.6.8.10}\lambda^5+\frac{1^2.5^2.9^2.25}{2.4.6\ldots14}\lambda^7+\&\text{c.}\right\}.$$

THE SOLUTION OF A SPECIAL CASE OF THE PROBLEM OF THE ESTABLISHMENT OF A CORRELATION BETWEEN TWO PLANE FIGURES (SECOND PAPER).

By *J. Brill, M.A.*, St. John's College.

1. In a paper published in the *Messenger of Mathematics* for August last under the above title,* I discussed the problem of the establishment of a correlation between the space outside a closed convex rectilinear polygon and a strip of a plane bounded by a finite straight line and two straight lines of infinite length at right angles to it. I now propose to adapt the method to the establishment of a correlation between the space included between two closed convex rectilinear polygons, one of which lies wholly within the other, and a rectangle.

2. Let $A_1A_2A_3 \dots A_m$ be the inner polygon, and let $\alpha_1, \alpha_2, \alpha_3, \dots, \alpha_m$ be the exterior angles between A_mA_1 and A_1A_2, A_1A_2 and A_2A_3, &c., respectively. Also let $B_1B_2B_3 \dots B_n$ be the outer polygon, and let $\beta_1, \beta_2, \beta_3, \dots, \beta_n$ be the exterior angles between B_nB_1 and B_1B_2, B_1B_2 and B_2B_3, &c., respectively



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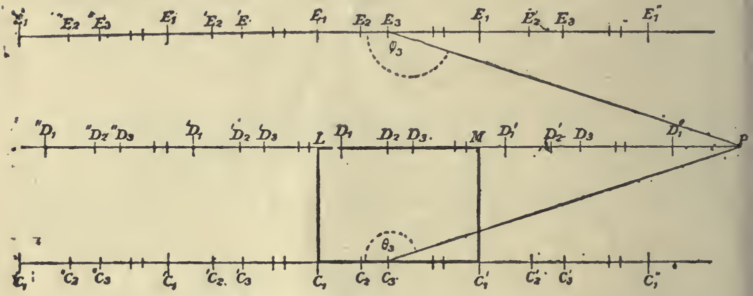
* *Vide* p. 57.

Take A_1 as origin, A_1A_2 as the axis of x, and a straight line at right angles to it as the axis of y.

Let $C_1C_1'ML$ be the rectangle with which the correlation is to be established, taking C_1C_1' as the axis of ξ and C_1 as origin. Also let C_1C_1' be the correlative of the perimeter $A_1A_2 \dots A_m$ of the inner polygon, C_1 and C_1' corresponding to A_1, and C_2, C_3, &c., corresponding to A_2, A_3, &c. Divide the axis of ξ into a series of portions $C_1'C_1''$, $C_1''C_1'''$, &c., $'C_1C_1$, $''C_1'C_1$, &c., each equal to C_1C_1', in a similar manner as was done on page 59. Also divide each of these portions into lengths equal to those into which the portion C_1C_1' is divided by the points C_2, C_3, &c., distinguishing the several points of division by subscripts and dashes as on page 59.

Draw a straight line of infinite length parallel to LM and C_1C_1', on the opposite side of LM to C_1C_1' and at the same distance from it. Divide this line into portions equal to those into which the axis of ξ is divided, each point of division being the foot of the perpendicular from the corresponding point of division of the axis of ξ on the line in question. Denote each of these points of division by the letter E with the subscripts and dashes of the point on the axis of ξ to which it corresponds.



We may now form an expression with regard to the C points similar to that we formed on page 60, viz.

$$\frac{1}{\pi}\Sigma\,(\alpha_1\theta_1 + \alpha_2\theta_2 + \dots + \alpha_m\theta_m).$$

If we discuss the values which this assumes along the axis of ξ, we find that they correspond to the values assumed around the perimeter of the inner polygon by the quantity we have denoted by ϑ in the previous paper. We may also form a similar expression with regard to the E points, replacing the symbol θ by the symbol ϕ, viz.

$$\frac{1}{\pi}\Sigma\,(\alpha_1\phi_1+\alpha_2\phi_2+\ldots+\alpha_m\phi_m).$$

The sum of these two expressions will have the value zero along the infinite straight line of which LM is a part. For, consider for example the portions contributed by C_3 and E_3, they give

$$\frac{1}{\pi}(\alpha_3\theta_3+\alpha_3\phi_3)=\frac{\alpha_3}{\pi}(\theta_3+\phi_3)=0,$$

since θ_3 and ϕ_3 are equal in numerical value but opposite in sign. We may thus consider the E system as the image of the C system in the indefinite line of which LM forms part. With the introduction of this image, the condition that should be satisfied along the axis of ξ is disturbed. We have therefore to take the image of the E system in the axis of ξ, and again to take the image of that in the indefinite line of which LM forms part, and so on indefinitely.

If now we take $C_1C_1'=f$, $C_1L=g$, and a_1, a_2, a_3, ..., a_m as the distances of the points C_1, C_2, C_3, ..., C_m from the origin of the correlative diagram, a_1 being introduced for the sake of symmetry although its value is zero; then the portion of $\log(d\zeta/dz)$ that arises from the point C_r and its images, and its corresponding points along the axis of ξ and their images, is

$$\begin{aligned}\frac{\alpha_r}{\pi}\Big\{&\log\sin\frac{\pi}{f}(\zeta-a_r)+\log\sin\frac{\pi}{f}(\zeta-a_r-2ig)\\&+\log\sin\frac{\pi}{f}(\zeta-a_r-4ig)+\ldots+\log\sin\frac{\pi}{f}(\zeta-a_r+2ig)\\&+\log\sin\frac{\pi}{f}(\zeta-a_r+4ig)+\ldots\Big\}\\=\frac{\alpha_r}{\pi}\log\Big\{&\sin\frac{\pi}{f}(\zeta-a_r)\sin\frac{\pi}{f}(\zeta-a_r-2ig)\sin\frac{\pi}{f}(\zeta-a_r+2ig)\\&\sin\frac{\pi}{f}(\zeta-a_r-4ig)\sin\frac{\pi}{f}(\zeta-a_r+4ig)\ldots\ldots ad.\ inf.\Big\}\\=\frac{\alpha_r}{\pi}\log H&\left\{\frac{2K}{f}(\zeta-a_r)\right\}+Q,\end{aligned}$$

where Q is an infinite constant, and we have $K'/K=2g/f$.

The portion of ϑ arising from this portion of $\log(d\zeta/dz)$ will satisfy the condition to be satisfied by ϑ along the perimeter of the inner polygon and will have the value zero along the perimeter of the outer polygon. We must therefore add to this portion of $\log(d\zeta/dz)$ a similar portion, such that the portion of ϑ arising from it satisfies the condition to be satisfied by ϑ along the perimeter of the outer polygon and has the value zero along the perimeter of the inner polygon. We proceed to obtain an expression for this second portion.

Let D_1D_1' be the correlative of the perimeter $B_1B_2B_3 \dots B_n$ of the outer polygon, D_1 and D_1' corresponding to B_1, and D_2, D_3, &c., corresponding to B_2, B_3, &c., D_1D_1' will be equal in length to LM and will be measured on the line of which LM forms a part, but D_1 and D_1' will not in general coincide with L and M. Divide the line of which LM forms a part into a series of portions $D_1'D_1''$, $D_1''D_1'''$, &c., ${}'D_1D_1$, ${}''D_1{}'D_1$, &c., each equal to D_1D_1'. Also divide each of these portions into lengths equal to those into which the portion D_1D_1' is divided by the points D_2, D_3, &c., distinguishing the several points of division by subscripts and dashes as above.

We may now write down an expression

$$\frac{1}{\pi}\Sigma\,(\beta_1\psi_1 + \beta_2\psi_2 + \dots + \beta_n\psi_n) - \gamma$$

with respect to the D points similar to those we have used above with respect to the C and E points, γ being the acute angle between the directions of A_1A_2 and B_1B_2, and ψ taking the place of the θ and ϕ of the former expressions. If we discuss the values assumed by this expression as we pass in a positive direction along the line of which LM forms a part and on the under side of it, we see that it assumes the same numerical values as ϑ does along the perimeter of the outer polygon but is of opposite sign. Thus the portion of $\log(d\zeta/dz)$ arising from the D points and their images is the sum of a series of terms of the form

$$-\frac{\beta_r}{\pi}\log H\left\{\frac{2K}{f}(\zeta - b_r - ig)\right\},$$

b_1, b_2 b_3, ..., b_n being the respective distances of the points D_1, D_2, D_3, ..., D_n from L, b_1 not being in general zero.

Thus we have

$$\frac{d\zeta}{dz} = R\,\frac{\left[H\left\{\frac{2K}{f}(\zeta - a_1)\right\}\right]^{\frac{\alpha_1}{\pi}} \dots \left[H\left\{\frac{2K}{f}(\zeta - a_m)\right\}\right]^{\frac{\alpha_m}{\pi}}}{\left[H\left\{\frac{2K}{f}(\zeta - b_1 - ig)\right\}\right]^{\frac{\beta_1}{\pi}} \dots \left[H\left\{\frac{2K}{f}(\zeta - b_n - ig)\right\}\right]^{\frac{\beta_n}{\pi}}},$$

where R is some finite constant* which may be complex.

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* We may obviously reject the infinite constant occurring above.

Therefore the correlation can be established with the aid of the equation

$$z = C\int d\zeta \frac{\left[H\left\{\frac{2K}{f}(\zeta - b_1 - ig)\right\}\right]^{\frac{\beta_1}{\pi}} \dots \left[H\left\{\frac{2K}{f}(\zeta - b_n - ig)\right\}\right]^{\frac{\beta_n}{\pi}}}{\left[H\left\{\frac{2K}{f}(\zeta - a_1)\right\}\right]^{\frac{\alpha_1}{\pi}} \dots \left[H\left\{\frac{2K}{f}(\zeta - a_m)\right\}\right]^{\frac{\alpha_m}{\pi}}},$$

where $C = 1/R$, and we have $K'/K = 2g/f$.

The remarks that were made in the former paper about the determination of the constants apply equally well to the case discussed in this.

ON A DIRECT RELATION BETWEEN THE DEFINITE ELLIPTIC INTEGRALS OF THE FIRST AND SECOND ORDER.

By the Rev. *G. F. Childe, M.A.*

(I). In the Theory of Elliptic Functions, corresponding to the modulus e, the first and second functions are here denoted by

$$F = \int_0^{\frac{1}{2}\pi} \frac{d\phi}{(1 - e^2 \sin^2\phi)^{\frac{1}{2}}};\quad E = \int_0^{\frac{1}{2}\pi} (1 - e^2 \sin^2\phi)^{\frac{1}{2}}\, d\phi;$$

and if any other modulus e_n is employed, F_n or E_n will indicate the functions.

It is now proposed to establish a direct relation between these two integrals; and, to this end, let it be supposed that they are subject to the condition

$$F = (1 + m)\, E \dots\dots\dots\dots (1),$$

m being some unknown function of e: it will be our first object to shew that m may be determined by the solution of a differential equation of the first order and degree.

Assuming the known equations

$$F = E - e\frac{dE}{de} \dots\dots\dots\dots (2),$$

$$E = (1 - e^2)\left(F + e\frac{dF}{de}\right) \dots\dots\dots\dots (3),$$

By combining (1), (2), we find $\frac{dE}{de} = -\frac{mE}{e}$; and (1) gives

$$\frac{dF}{de} = E\frac{dm}{de} + (1+m)\frac{dE}{de};$$

therefore $$e\frac{dF}{de} = eE\frac{dm}{de} - m(1+m)E.$$

By (3) $$e\frac{dF}{de} = \frac{E}{1-e^2} - (1+m)E,$$

$$\frac{E}{1-e^2} - (1+m)E = eE\frac{dm}{de} - m(1+m)E;$$

therefore $$e\frac{dm}{de} - m^2 = \frac{e^2}{1-e^2} \quad \text{..................} \quad (4).$$

The value of m obtained from this equation, which is similar in form to Riccati's, determines the relation between E and F expressed by (1); or, as it may be written,

$$F : E :: (1+m) : 1.$$

By the relation (1) the two functions E and F are reciprocally convertible; so that any expression in terms of E may be transformed into a corresponding expression for F, and conversely.

Hence, further, any equation of the form

$$f(E, F) = 0$$

implies two separate equations

$$f\{E, (1+m)E\} = 0, \text{ and } f\left\{F, \frac{F}{1+m}\right\} = 0.$$

NOTE. Since each of these elliptic function becomes circular when $e = 0$; therefore

$$E = \tfrac{1}{2}\pi = F;$$

consequently, by (1), in correcting the integral of (4) there is the condition $e = 0$, $m = 0$, simultaneously.

(II). The equation $e\frac{dm}{de} - m^2 = \frac{e^2}{1-e^2}$ may be integrated by continued approximation, in the manner following;

First, let m^2 be omitted, then

$$\frac{dm}{de} = \frac{e}{1-e^2};$$

therefore $$m = \log\frac{1}{(1-e^2)^{\frac{1}{2}}} + C.$$

Since m and e vanish together, $C=0$; and, disregarding the fourth and higher powers of e,

$$m = \log\frac{1}{1-\frac{1}{2}e^2} = \frac{e^2}{2-e^2}, \text{ (approximately)}\ldots\ldots(1).$$

For a second approximation, assume

$$m = \frac{e^2}{2-e^2} + k \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2),$$

then

$$\frac{dm}{de} = \frac{4e}{(2-e^2)^2} + \frac{dk}{de},$$

$$m^2 = \frac{e^4}{(2-e^2)^2} + \frac{2e^2k}{2-e^2} + k^2,$$

When m^2 and $\frac{dm}{de}$ have been eliminated from the original equation the differential equation for k becomes, after the requisite reduction,

$$(2-e^2)\frac{dk}{de} - 2ek = \frac{e}{1-e^2} - \frac{2e}{2-e^2} + \frac{2-e^2}{e}k^2\ldots\ldots(3).$$

In order to obtain a first approximate value of k, let k^2 be omitted in (3); then, after correction of the integral by the condition $k=0$ when $e=0$, we find

$$(2-e^2)\,k = \log\frac{2-e^2}{2(1-e^2)^{\frac{1}{2}}};$$

therefore

$$k = \frac{1}{2-e^2}\log\frac{2-e^2}{2(1-e^2)^{\frac{1}{2}}}\ldots\ldots\ldots\ldots\ldots\ldots(4),$$

as a first approximation to k.

For a second approximation to k we now substitute for k^2 in (3) its value as determined in (4).

Let

$$v = \frac{1}{2^2}\,\frac{e^4}{1-e^2};\quad v^n = \frac{1}{2^{2n}}\,\frac{e^{4n}}{(1-e^2)^n};$$

therefore

$$(2-e^2)\,k = \tfrac{1}{2}\log(1+v),$$

$$2^2(2-e^2)^2k^2 = A_2v^2 - A_3v^3 + A_4v^4 - \ldots + (-1)^n A_n v^n\ldots(5).$$

The coefficients A_2, A_3, ..., A_n, will be afterwards determined, and, when the proper substitutions have been made, the equation will be obtained

$$\frac{2-e^2}{e}k^2 = \frac{1}{2-e^2}\left\{\frac{A_2}{2^6}\,\frac{e^7}{(1-e^2)^2} - \frac{A_3}{2^8}\,\frac{e^{11}}{(1-e^2)^3} + \ldots \pm \frac{A_{n+1}}{2^{2n+4}}\,\frac{e^{4n+3}}{(1-e^2)^{n+1}}\right\}.$$

Let
$$C_n = \frac{A_{n+1}}{2^{2n+4}};$$

therefore
$$\int \frac{2-e^2}{e} k^2 de = C_1 \int \frac{e^7 de}{(2-e^2)(1-e^2)^2} - \ldots \pm C_n \int \frac{e^{4n+3} de}{(2-e^2)(1-e^2)^{n+1}}.$$

Writing
$$V_n = \int_0^e \frac{e^{4n+3} de}{(2-e^2)(1-e^2)^{n+1}},$$

an integration of (3) is thus effected by the equation
$$(2-e^2)k = \log \frac{2-e^2}{2(1-e^2)^{\frac{1}{2}}} + C_1 V_1 - C_2 V_2 + \ldots \pm C_n V_n \ldots (6),$$

in which the series is capable of indefinite extension. Since
$$\frac{e^{4n+3}}{(2-e^2)(1-e^2)^{n+1}} + \frac{4e^{4n-1}}{(2-e^2)(1-e^2)^n} = \frac{(2-e^2)e^{4n-1}}{(1-e^2)^{n+1}}$$
$$= \frac{1}{2n} \frac{d}{de} \left(\frac{e^4}{1-e^2}\right)^n;$$

therefore
$$V_n + 4V_{n-1} = \frac{1}{2n} \left(\frac{e^4}{1-e^2}\right)^n \ldots\ldots\ldots\ldots\ldots (7).$$

The correction may be made on the fundamental integral, when $n=1$, and
$$V_1 + 4V_0 = \tfrac{1}{2} \frac{e^4}{1-e^2},$$

but
$$V_0 = \int_0^e \frac{e^3 de}{(2-e^2)(1-e^2)} = \log \frac{2-e^2}{2(1-e^2)^{\frac{1}{2}}}.$$

Thus the integrals in (6) are successively determined by the following equations:
$$V_0 = \log \frac{2-e^2}{2(1-e^2)^{\frac{1}{2}}},$$
$$V_1 = \tfrac{1}{2} \left(\frac{e^4}{1-e^2}\right)^1 - 4V_0,$$
$$V_2 = \tfrac{1}{4} \left(\frac{e^4}{1-e^2}\right)^2 - 4V_1,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,$$
$$V_n = \frac{1}{2n} \left(\frac{e^4}{1-e^2}\right)^n - 4V_{n-1} \ldots\ldots\ldots\ldots\ldots (8).$$

Finally, k has the approximate value,

$$k = \frac{1}{2-e^2}\left(V_0 + C_1 V_1 - C_2 V_2 + \ldots \pm C_n V_n\right) \;\ldots\ldots(9),$$

and m is given by the equation (2) of this section.

COR. The general expansion of V_n is

$$V_n = \frac{1}{2n}\left(\frac{e^4}{1-e^2}\right)^n - \frac{2^2}{2(n-1)}\left(\frac{e^4}{1-e^2}\right)^{n-1} + \frac{2^4}{2(n-2)}\left(\frac{e^4}{1-e^2}\right)^{n-2} - \ldots \pm 2^{2n} V_0,$$

in which there are $n+1$ terms, the last term being $\left.\begin{matrix}+\\-\end{matrix}\right\}$ as n is $\left\{\begin{matrix}\text{even}\\\text{odd}\end{matrix}\right.$.

It remains to ascertain the numerical values of the coefficients in the equations (5) and (9), and their law of derivation.

Taking any odd number of terms in (5), so that $A_n v^n$ is positive,*

$$\{\log(1+v)\}^2 = A_2 v^2 - A_3 v^3 + \ldots - A_{n-1} v^{n-1} + A_n v^n,$$

$$\frac{2\log(1+v)}{1+v} = 2A_2 v - 3A_3 v^2 + \ldots - (n-1)A_{n-1} v^{n-2} + nA_n v^{n-1}.$$

After expanding $\log(1+v)$, multiplying by $(1+v)$, and equating coefficients of the same powers of v in the two equivalent series, we find

$$nA_n - (n-1)A_{n-1} = \frac{2}{n-1};\ A_2 = 1 \;\ldots\ldots\ldots(10).$$

This expression gives the consecutive coefficients with much facility, and the general value is

$$A_n = \frac{2}{n}\left(\tfrac{1}{1} + \tfrac{1}{2} + \tfrac{1}{3} + \ldots + \frac{1}{n-1}\right) \;\ldots\ldots\ldots\ldots(11).$$

Again, $C_n = \dfrac{A_{n+1}}{2^{2n+4}}$; $C_{n-1} = \dfrac{A_n}{2^{2n+2}}$; hence we obtain the derivative form

$$n(n+1)\,C_n = (\tfrac{1}{2}n)^2\, C_{n-1} + \frac{1}{2^{2n+3}} \;\ldots\ldots\ldots\ldots(12).$$

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* This is immaterial, but the sign should be defined.

COR. From (II) it appears, conversely, that the sum of $n-1$ terms of the reciprocals of natural numbers is equal to $\frac{1}{2}n$ times the coefficient of x^n in the expansion of the function $\{\log(1+x)\}^2$.

NOTE. The series (9) will give k, and, consequently, F correctly as far as $e = \cdot 65$ (nearly), to *seven* decimals; but to a less degree of accuracy when e exceeds this value.

It will, however, be shewn that this defect is remedied by the formula given in section (III), (*c*), equation (13).

(III). Some applications will now be given of the formula $F=(1+m)E$, which has been here established. When several moduli are employed, $e, e_1, e_2, \dots, e_n$, unless the contrary is stated, they will be connected by the relation

$$(1+e_n)\,e_{n-1} = 2\sqrt{e_n},$$

in which $e_0 = e$.

(*a*) Taking first the two remarkable expressions of Legendre for F and E,

$$F = \tfrac{1}{2}\pi(1+e_1)\dots(1+e_n) \;\dots\dots\dots\dots\dots\; (1),$$

$$E = \tfrac{1}{2}\pi\frac{2^n}{e}\left\{\frac{e_n}{e_1\dots e_{n-1}}\right\}(\Sigma) \;\dots\dots\dots\dots\dots\dots (2),$$

we have the corresponding expressions for E and F

$$E = \frac{\pi}{2(1+m)}(1+e_1)\dots(1+e_n) \;\dots\dots\dots\dots (3),$$

$$F = \tfrac{1}{2}\pi(1+m)\frac{2^n}{e}\left\{\frac{e_n}{e_1\dots e_{n-1}}\right\}(\Sigma) \dots\dots\dots\dots (4),$$

where $(\Sigma) = 1 - \tfrac{1}{2}e^2\left(1+\tfrac{1}{2}e_1 + \tfrac{1}{4}e_1e_2 + \dots + \frac{1}{2^{n-1}}e_1\dots e_{n-1}\right)$.

(*b*) When F_1 has been eliminated between the known equations

$$(1-e_1^2)F_1 = 2E_1 - (1+e_1)E \;\dots\dots\dots\dots (5),$$

$$F = (1+e_1)F_1 \;\dots\dots\dots\dots\dots\dots\dots (6),$$

it will be found that

$$F = E + \frac{2(E_1 - E)}{1-e_1} \dots\dots\dots\dots\dots\dots (7);$$

consequently $$mE = \frac{2(E_1-E)}{1-e_1} \dots\dots\dots\dots\dots\dots\dots\dots (8).$$

From the last equation there is

$$E_1 = \left(1 + \frac{1-e_1}{2} m\right) E \text{ .... } (9).$$

Hence

$$E_2 = \left(1 + \frac{1-e_2}{2} m_1\right) E_1 = \left(1 + \frac{1-e_1}{2} m\right) \left(1 + \frac{1-e_2}{2} m_1\right) E,$$

and, generally,

$$E_n = \left(1 + \frac{1-e_1}{2} m\right) \left(1 + \frac{1-e_2}{2} m_1\right) \dots \left(1 + \frac{1-e_n}{2} m_{n-1}\right) E \text{ .... } (10).$$

Now as the factorial is indefinitely extended, e_n and m_n continually diminish, while $E_n = \frac{1}{2}\pi$ ultimately, but E remains unchanged; therefore

$$\tfrac{1}{2}\pi = \left(1 + \frac{1-e_1}{2} m\right) \dots \left(1 + \frac{1-e_n}{2} m_{n-1}\right) E \dots (11).$$

If M is written for the factorial expression, $\dfrac{2\pi a}{4aE} = M$; and this is the ratio of the circumference of the circle of radius a, circumscribing an ellipse of any eccentricity e, to the periphery of that curve. Also, by (11),

$$E = \tfrac{1}{2}\pi \frac{1}{M} \text{ and } F = \tfrac{1}{2}\pi \frac{1+m}{M} \text{ .... } (12).$$

(*c*) We may now determine a relation between m depending on e, and m_1 depending on e_1; e_1, m_1 being less than e, m respectively, and $(1 + e_1) e = 2\sqrt{e_1}$.

Since $\qquad F = (1 + e_1) F_1$,

therefore $\qquad (1 + m) E = (1 + e_1)(1 + m_1) E_1$,

$$(1 + m) = (1 + e_1)(1 + m_1)\left(1 + \frac{1-e_1}{2} m\right),$$

by (9). Hence

$$\tfrac{1}{2}m = \frac{e_1 + (1 + e_1) m_1}{(1 + e_1^2) - (1 - e_1^2) m_1} \text{ .... } (13).$$

Since the formal relation between m_1 and m_2, ..., m_{n-1} and m_n, must be the same as between m and m_1, there will be the general expression

$$\tfrac{1}{2}m_{n-1} = \frac{e_n + (1 + e_n) m_n}{(1 + e_n^2) - (1 - e_n^2) m_n} \text{ .... } (14).$$

Thus m can always be brought to depend on m_n, which, by carrying the process far enough, may be made as small as we please.

The expression (13) is equivalent to

$$(1+m)=\frac{(1+e_1)^2(1+m_1)}{(1+e_1^2)-(1-e_1^2)\,m_1}=\frac{(1+e_1)^2(1+m_1)}{(1-m_1)+e_1^2(1+m_1)}\ldots(15),$$

to which the remark as to generality of course applies.

In section (II) it was remarked that the series (9), as far as $e=\cdot65$, will give F correctly to *seven* decimals; but to a less degree of accuracy when e is greater than this.

By the formulæ (13), (14) m can always be made to depend on m_n, which (e_n being made less than $\cdot65$) may be determined by (9). Thus that series is rendered available with equal accuracy for *all* values of the modulus. If $e=\cdot9999$, m_2 is sufficient; e_2 being then $=\cdot62$ nearly.

(d) Differentiating $F=(1+m)\,E$,

$$\frac{dF}{de}=E\frac{dm}{de}+(1+m)\frac{dE}{de}.$$

But $$\frac{dm}{de}=\frac{e}{1-e^2}+\frac{m^2}{e},\quad \frac{dE}{de}=-\frac{mE}{e}.$$

Hence $$\frac{dF}{de}=\left(\frac{e}{1-e^2}-\frac{m}{e}\right)E\ldots\ldots\ldots\ldots\ldots(16).$$

From this, since $$-\frac{mE}{e}=\frac{dE}{de},$$

we obtain $$\frac{d}{de}(F-E)=\frac{Ee}{1-e^2}.$$

Therefore $$(F-E)=mE=\int_0^e\frac{E\,e\,de}{1-e^2}\ldots\ldots\ldots\ldots\ldots(17).$$

Therefore, by (b), (8),

$$\int_0^e\frac{E\,e\,de}{1-e^2}=\frac{2\,(E_1-E)}{1-e_1}.$$

After substitution of the usual expansion for E under the integral sign in (17), the resulting series is very readily integrated; though it is then but slowly convergent, except for low values of the modulus.

The series for mE is thus found to be

$$mE = \tfrac{1}{2}\pi\,\{1 - (C_2 + C_4 + \ldots + C_{2n})\} \log \frac{1}{(1-e^2)^{\frac{1}{2}}}$$
$$+ \tfrac{1}{2}\pi \left\{ C_2\,(\tfrac{1}{2}e^2) + C_4\,(\tfrac{1}{2}e^2 + \tfrac{1}{4}e^4) + \ldots + C_{2n} \left(\tfrac{1}{2}e^2 + \ldots + \frac{e^{2n}}{2n} \right) \right\} \qquad \ldots\ldots\ldots\ldots(18),$$

in which $$C_{2n} = \frac{1}{2n-1} \left\{ \frac{1.3\ldots(2n-1)}{2.4\ldots 2n} \right\}^2.$$

(*e*) When e and e_1 are connected by the equation $e^2 + e_1^2 = 1$, we have the known relation

$$EE_1 - (F - E)(F_1 - E_1) = \tfrac{1}{2}\pi.$$

Eliminating F and F_1,

$$EE_1 = \tfrac{1}{2}\pi \frac{1}{1 - mm_1}, \quad FF_1 = \tfrac{1}{2}\pi \frac{(1+m)(1+m_1)}{1 - mm_1} \ldots(19).$$

COR. When $e_1 = e = \dfrac{1}{\sqrt{2}}$, these expressions become

$$E = \left(\tfrac{1}{2}\pi \frac{1}{1-m^2} \right)^{\frac{1}{2}}, \quad F = \left(\tfrac{1}{2}\pi \frac{1+m}{1-m} \right)^{\frac{1}{2}};$$

and therefore $$EF = \frac{\pi}{2(1-m)}.$$

These examples may suffice in some degree to illustrate the effect of the relation $F = (1 + m)\,E$ in modifying or extending theorems connecting the first and second elliptic integrals. Moreover, since complete functions of the third order can be reduced to dependence on the two former, it follows that elliptic functions of any class may be made to depend ultimately on the single function E, which represents the rectification of the elliptic quadrant.

Thus it appears that a fundamental table, of such extent as to give E for all values of e, is sufficient to effect the final solution of all problems which depend upon these definite integrals. And with this view a table for E has been constructed by the writer of this paper, embracing values of e from $e = {\cdot}0000$, when the quadrant of the ellipse is hardly distinguishable from that of the circumscribed circle, to $e = {\cdot}999999999$, when it differs in length by a quantity almost inappreciable from its semi-axis major. This table consists of about two thousand two hundred terms, independently computed to seven decimals by appropriate series, with an increment of e sufficiently small to secure accuracy in interpolation; and by its use, combined with the formulæ

$$F=(1+m)E, \quad \text{or} \quad F=E+\frac{2(E_1-E)}{1-e_1},$$

the first integral F, for any value of e, may be determined without the necessity for an independent tabular development of that function.

ON THE EXPANSION OF $\frac{G}{K}$, $\frac{K}{G}$, $\frac{I}{E}$, &c., IN ASCENDING POWERS OF k^2.

By *J. W. L. Glaisher.*

The object of the present paper is to consider the developments of $\frac{G}{K}$, $\frac{K}{G}$, $\frac{I}{E}$, $\frac{E}{I}$, and of certain other closely connected quantities, in ascending powers of k^2.

Expansion of $\frac{G}{K}$, §§ 1–4.

§ 1. The development of $\frac{E}{K}$ in ascending powers of k was considered by Gudermann in § 94 of his *Theorie der Modular-Functionen und der Modular-Integrale.*[*]

Putting $$\frac{E}{K}=1-t,$$

Gudermann found that t satisfied the differential equation

$$\frac{dt}{dk}=\frac{(1-2t)k}{k'^2}+\frac{t^2}{kk'^2}$$

Replacing t by $\frac{1}{2}k^2v$, and putting $x=\frac{1}{4}k^2$, he transformed the differential equation into

$$(1-4x)\frac{dv}{dx}+\frac{v-1}{x}-v^2=0.$$

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[*] p. 180; or *Crelle's Journal*, vol. XVIII., p. 356.

Putting $v=1+a_1x+a_2x^2+a_3x^3+a_4x^4+\&c.$, substituting in the differential equation, and equating coefficients, Gudermann found

$$a_1=\frac{1}{2}, \qquad a_5=\frac{727}{2^5},$$

$$a_2=1, \qquad a_6=\frac{1171}{2^4},$$

$$a_3=\frac{41}{2^4}, \qquad a_7=\frac{498409}{2^{11}},$$

$$a_4=\frac{59}{2^3}, \qquad a_8=\frac{848479}{2^{10}};$$

whence

$$t=\frac{k^2}{2}+\frac{k^4}{2^4}+\frac{k^6}{2^5}+\frac{41k^8}{2^{11}}+\&c.$$

Gudermann calls attention to the fact that all the denominators are powers of 2.

§ 2. Gudermann's t is equal to $\frac{K-E}{K}=-\frac{I}{K}$; so that, putting $h=k^2$, his series may be written

$$\frac{I}{K}=-\frac{h}{2}-\frac{h^2}{2^4}-\frac{h^3}{2^5}-\frac{41h^4}{2^{11}}-\&c.$$

Since $G=I+hK$, we find by adding h to each side of the equation,

$$\frac{G}{K}=\frac{h}{2}-\frac{h^2}{2^4}-\frac{h^3}{2^5}-\frac{41h^4}{2^{11}}-\&c.,$$

or, using the coefficients a_1, a_2, ..., whose values are given above,

$$\frac{2G}{K}=h-a_1\frac{h^2}{4}-a_2\frac{h^3}{4^2}-a_3\frac{h^4}{4^3}-a_4\frac{h^5}{4^4}-\&c.$$

§ 3. We may obtain the above formula very simply by the following process, which differs from Gudermann's only in detail.

Let $u=\frac{2G}{K}$; then, since

$$\frac{dK}{dh}=\frac{G}{2hh'}, \qquad \frac{dG}{dh}=\frac{K}{2},$$

we have

$$\frac{du}{dh} = \frac{2}{K^2}\left\{K\frac{K}{2} - G\frac{G}{2hh'}\right\}$$

$$= 1 - \frac{1}{hh'}\frac{G^2}{K^2} = 1 - \frac{u^2}{4hh'},$$

so that
$$hh'\left(1 - \frac{du}{dh}\right) = \frac{u^2}{4}.$$

Now, let $u = h - a_1\frac{h^2}{4} - a_2\frac{h^3}{4^2} - a_3\frac{h^4}{4^3} - \&c.;$

then

$$(h - h^2)\left(1 - \frac{du}{dh}\right) = 2a_1\frac{h^2}{4} + (3a_2 - 8a_1)\frac{h^3}{4^2} + (4a_3 - 12a_2)\frac{h^4}{4^3} + \&c.,$$

and

$$\frac{u^2}{4} = \frac{h^2}{4} - 2a_1\frac{h^3}{4^2} - (2a_2 - a_1^2)\frac{h^4}{4^3} - (2a_3 - 2a_2a_1)\frac{h^5}{4^4} - \&c.,$$

whence, by equating coefficients,

$$2a_1 = 1,$$
$$3a_2 = 6a_1,$$
$$4a_3 = 10a_2 + a_1^2,$$
$$5a_4 = 14a_3 + 2a_2a_1,$$
$$6a_5 = 18a_4 + 2a_3a_1 + a_2^2,$$
$$\&c., \qquad \&c.$$

These are the same as Gudermann's equations, from which he obtained the values of a_1, a_2, &c., and on solving them I find $a_1 = \frac{1}{2}$, $a_2 = 1$, $a_3 = \frac{41}{16}$, $a_4 = \frac{59}{8}$, $a_5 = \frac{727}{32}$, agreeing with Gudermann's results.

§ 4. We may, however, also obtain the series for $\frac{2G}{K}$ as the quotient of the two series

$$\frac{2K}{\pi} = 1 + \frac{1^2}{2^2}h + \frac{1^2.3^2}{2^2.4^2}h^2 + \frac{1^2.3^2.5^2}{2^2.4^2.6^2}h^3 + \&c.,$$

$$\frac{2G}{\pi} = \frac{1}{2}h + \frac{1^2}{2^2.4}h^2 + \frac{1^2.3^2}{2^2.4^2.6}h^3 + \&c.$$

For, putting as before

$$\frac{2G}{K} = h - a_1 \frac{h^2}{4} - a_2 \frac{h^3}{4^2} - \&c.,$$

and equating coefficients in the formula

$$\left\{\frac{h}{4} + \frac{1^2}{1^2}\frac{1}{2}\frac{h^2}{4^2} + \frac{1^2.3^2}{1^2.2^2}\frac{1}{3}\frac{h^3}{4^3} + \frac{1^2.3^2.5^2}{1^2.2^2.3^2}\frac{1}{4}\frac{h^4}{4^4} + \&c.\right\}$$

$$= \left\{1 + \frac{h}{4} + \frac{1^2.3^2}{1^2.2^2}\frac{h^2}{4^2} + \frac{1^2.3^2.5^2}{1^2.2^2.3^2}\frac{h^3}{4^3} + \&c.\right\}\left\{\frac{h}{4} - a_1\frac{h^2}{4^2} - a_2\frac{h^3}{4^3} - \&c.\right\},$$

we obtain the following system of equations

$$a_1 = \frac{1}{2},$$

$$a_2 + a_1 = \frac{1^2.3^2}{1^2.2.3},$$

$$a_3 + a_2 + \frac{1^2.3^2}{1^2.2^2}a_1 = \frac{1^2.3^2.5^2}{1^2.2^2.3.4},$$

$$a_4 + a_3 + \frac{1^2.3^2}{1^2.2^2}a_2 + \frac{1^2.3^2.5^2}{1^2.2^2.3^2}a_1 = \frac{1^2.3^2.5^2.7^2}{1^2.2^2.3^2.4.5},$$

$$a_5 + a_4 + \frac{1^2.3^2}{1^2.2^2}a_3 + \frac{1^2.3^2.5^2}{1^2.2^2.3^2}a_2 + \frac{1^2.3^2.5^2.7^2}{1^2.2^2.3^2.4^2}a_1 = \frac{1^2.3^2.5^2.7^2.9^2}{1^2.2^2.3^2.4^2.5.6},$$

&c., &c.

From these equations we find $a_1 = \frac{1}{2}$, $a_2 = 1$, $a_3 = \frac{41}{16}$, &c. as before. They do not give the values of a_1, a_2, ... quite as readily as those employed in the last section; but they afford a reason for the denominators consisting only of powers of 2; for in the above system of equations all the uneven divisors occurring in the denominators are cancelled by the divisors which have already appeared in the numerators, so that only powers of 2 are left in the denominators.*

Expansion of log K, § 5, 6.

§ 5. We may deduce the expansion of $\log K$ from that of $\frac{2G}{K}$. For

$$\frac{2G}{K} = \frac{4hh'}{K}\frac{dK}{dh} = 4hh'\frac{d}{dh}\log K,$$

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* This is true also of the expansion of $\frac{2K}{\pi}$ in ascending powers of h, in which the denominators consist only of powers of 2.

whence $h'\dfrac{d}{dh}\log K = \dfrac{1}{4} - a_1\dfrac{h}{4^2} - a_2\dfrac{h^2}{4^3} - a_3\dfrac{h^3}{4^4} - \text{\&c.},$

and therefore, multiplying by $1 + h + h^2 + h^3 + \text{\&c.}$,

$$\frac{dK}{dh} = \frac{1}{4} + \alpha_1\frac{h}{4^2} + \alpha_2\frac{h^2}{4^3} + \alpha_3\frac{h^3}{4^4} + \text{\&c.},$$

where
$$\begin{aligned}\alpha_1 &= 4 - a_1 = \tfrac{7}{2},\\ \alpha_2 &= 4^2 - 4a_1 - a_2 = 13,\\ \alpha_3 &= 4^3 - 4^2a_1 - 4a_2 - a_3 = \tfrac{791}{16},\\ &\text{\&c.,} \qquad\qquad \text{\&c.}\end{aligned}$$

Integrating, we have

$$\log K = \frac{h}{4} + \frac{\alpha_1}{2}\frac{h^2}{4^2} + \frac{\alpha_2}{3}\frac{h^3}{4^3} + \frac{\alpha_3}{4}\frac{h^4}{4^4} + \text{\&c.}$$

§ 6. We may also obtain the development of $\log K$ directly by equating coefficients as in § 4.

We have

$$\frac{\dfrac{dK}{dh}}{K} = \frac{\dfrac{1}{4} + \dfrac{1^2.3^2}{1^2.2}\dfrac{h}{4^2} + \dfrac{1^2.3^2.5^2}{1^2.2^2.3}\dfrac{h^2}{4^3} + \text{\&c.}}{1 + \dfrac{h}{4} + \dfrac{1^2.3^2}{1^2.2^2}\dfrac{h^2}{4^2} + \dfrac{1^2.3^2.5^2}{1^2.2^2.3^2}\dfrac{h^3}{4^3} + \text{\&c.}},$$

and, by assuming that this quotient

$$= \frac{1}{4} + \alpha_1\frac{h}{4^2} + \alpha_2\frac{h^2}{4^3} + \alpha_3\frac{h^3}{4^4} + \text{\&c.},$$

and equating coefficients, we find

$$\alpha_1 + 1 = \frac{1^2.3^2}{1^2.2},$$

$$\alpha_2 + \alpha_1 + \frac{1^2.3^2}{1^2.2^2} = \frac{1^2.3^2.5^2}{1^2.2^2.3},$$

$$\alpha_3 + \alpha_2 + \frac{1^2.3^2}{1^2.2^2}\alpha_1 + \frac{1^2.3^2.5^2}{1^2.2^2.3^2} = \frac{1^2.3^2.5^2.7^2}{1^2.2^2.3^2.4},$$

$$\text{\&c.,} \qquad\qquad \text{\&c.,}$$

giving $\alpha_1 = \frac{7}{2}$, $\alpha_2 = 13$, $\alpha_3 = \frac{791}{16}$, &c., as before.

Expansion of $\frac{G^2}{K^2}$, § 7.

§ 7. By means of the differential equation satisfied by $\frac{2G}{K}$ we may readily derive the expansion of $\frac{G^2}{K^2}$ from that of $\frac{2G}{K}$; for, u denoting $\frac{2G}{K}$ as in § 3,

$$\frac{G^2}{K^2} = hh'\left(1 - \frac{du}{dh}\right)$$

$$= 2a_1\frac{h^2}{4} + (3a_2 - 8a_1)\frac{h^3}{4^2} + (4a_3 - 12a_2)\frac{h^4}{4^3} + \&c.$$

$$= \frac{h^2}{2^2} - \frac{h^3}{2^4} - \frac{7h^4}{2^8} - \frac{33h^5}{2^{11}} - \&c.$$

Expansions of $\frac{K}{G}$, $\log K$, *and* $\frac{K^2}{G^2}$, §§ 8–11.

§ 8. If
$$u = \frac{2K}{G},$$

then we find at once, as in § 3,

$$\frac{du}{dh} = \frac{1}{hh'} - \frac{u^2}{4}.$$

Now, let

$$u = \frac{4}{h} + b_1 + b_2\frac{h}{4} + b_3\frac{h^2}{4^2} + b_4\frac{h^3}{4^3} + \&c.,$$

then, substituting in the differential equation, we have

$$-\frac{4}{h^2} + b_2\frac{1}{4} + 2b_3\frac{h}{4^2} + 3b_4\frac{h^2}{4^3} + \&c.$$

$$= \quad \frac{1}{h} + 1 + h + h^2 + h^3 + \&c.$$

$$-\frac{4}{h^2} - 2b_1\frac{1}{h} - (2b_2 + b_1{}^2)\frac{1}{4} - (2b_3 + 2b_2b_1)\frac{h}{4^2} - \&c.,$$

whence, equating coefficients, we obtain the system of equations

$$\begin{aligned} 2b_1 &= 1, \\ 3b_2 + b_1^2 &= 4, \\ 4b_3 + 2b_2b_1 &= 4^2, \\ 5b_4 + 2b_3b_1 + b_2^2 &= 4^3, \\ 6b_5 + 2b_4b_1 + 2b_3b_2 &= 4^4, \\ \&c., \qquad & \quad \&c., \end{aligned}$$

giving $b_1 = \frac{1}{2}$, $b_2 = \frac{5}{4}$, $b_3 = \frac{59}{16}$, $b_4 = \frac{47}{4}$, &c.

§ 9. By proceeding as in § 4, we have

$$\frac{2K}{G} = \frac{1 + \frac{h}{4} + \frac{1^2.3^2}{1^2.2^2}\frac{h^2}{4^2} + \frac{1^2.3^2.5^2}{1^2.2^2.3^2}\frac{h^3}{4^3} + \&c.}{\frac{h}{4} + \frac{1}{2}\frac{h^2}{4^2} + \frac{1^2.3^2}{1^2.2^2}\frac{1}{3}\frac{h^3}{4^3} + \frac{1^2.3^2.5^2}{1^2.2^2.3^2}\frac{1}{4}\frac{h^4}{4^4} + \&c.}$$

$$= \frac{4}{h} + b_1 + b_2\frac{h}{4} + b_3\frac{h^2}{4^2} + b_4\frac{h^3}{4^3} + \&c.,$$

whence, by equating coefficients, we obtain the system of equations

$$\begin{aligned} & b_1 = \frac{1}{2}, \\ & b_2 + \frac{b_1}{2} = \frac{1^2.3^2}{1^2.2.3}, \\ & b_3 + \frac{b_2}{2} + \frac{1^2.3^2}{1^2.2^2}\frac{b_1}{3} = \frac{1^2.3^2.5^2}{1^2.2^2.3.4}, \\ & b_4 + \frac{b_3}{2} + \frac{1^2.3^2}{1^2.2^2}\frac{b_2}{3} + \frac{1^2.3^2.5^2}{1^2.2^2.3^2}\frac{b_1}{4} = \frac{1^2.3^2.5^2.7^2}{1^2.2^2.3^2.4.5}, \\ & \qquad \&c., \qquad\qquad\qquad\qquad \&c., \end{aligned}$$

giving the same values of b_1, b_2, b_3, ..., as before. These equations also show that the denominators of these coefficients will consist only of powers of 2.

§ 10. Since

$$\frac{2K}{G} = \frac{4}{G}\frac{dG}{dh} = 4\frac{d}{dh}\log G,$$

we can pass immediately, by integration with respect to h, to the expansion of $\log G$.

We thus find

$$\log G = \log h + b_1 \frac{h}{4} + \frac{b_2}{2}\frac{h^2}{4^2} + \frac{b_3}{3}\frac{h^3}{4^3} + \frac{b_4}{4}\frac{h^4}{4^4} + \&c.,$$

where b_1, b_2, ... have the same values as in the last two sections.

§ 11. We have also

$$\frac{K^2}{G^2} = \frac{4}{h^2} + \frac{1}{h} + (4 - b_2)\frac{1}{4} + (4^2 - 2b_3)\frac{h}{4^2} + (4^3 - 3b_4)\frac{h^2}{4^3}$$

$$+ (4^4 - 4b_5)\frac{h^3}{4^4} + \&c.$$

$$= \frac{4}{h^2} + \frac{1}{h} + \frac{11}{2^4} + \frac{69h}{2^7} + \frac{115h^2}{2^8} + \&c.$$

Expansions of $\frac{I}{E}$ *and* $\frac{E}{I}$, §§ 12–15.

§ 12. Let

$$u = -\frac{2I}{E},$$

then, since

$$\frac{dI}{dh} = -\frac{E}{2h'}, \quad \frac{dE}{dh} = \frac{I}{2h},$$

we find at once

$$\frac{du}{dh} = \frac{1}{h'} + \frac{1}{h}\frac{u^2}{4}.$$

Putting

$$u = h + c_1\frac{h^2}{4} + c_2\frac{h^3}{4^2} + c_3\frac{h^4}{4^3} + \&c.$$

and substituting in the differential equation, we find

$$\begin{aligned} 2c_1 &= 1 + 4, \\ 3c_2 &= 2c_1 + 4^2, \\ 4c_3 &= 2c_2 + c_1{}^2 + 4^3, \\ 5c_4 &= 2c_3 + 2c_2c_1 + 4^4, \\ \&c., &\qquad \&c., \end{aligned}$$

giving $c_1 = \frac{5}{2}$, $c_2 = 7$, $c_3 = \frac{337}{16}$, $c_4 = \frac{533}{8}$, &c.

§ 13. If we derive the expansion from the quotient of the series

$$-\frac{4I}{\pi}=h+\frac{1^2}{1^2}\frac{3}{2}\frac{h^2}{4}+\frac{1^2.3^2}{1^2.2^2}\frac{5}{3}\frac{h^3}{4^2}+\frac{1^2.3^2.5^2}{1^2.2^2.3^2}\frac{7}{4}\frac{h^4}{4^3}+\&c.,$$

$$\frac{2E}{\pi}=1-\frac{h}{4}-\frac{1^2.3}{1^2.2^2}\frac{h^2}{4^2}-\frac{1^2.3^2.5}{1^2.2^2.3^2}\frac{h^3}{4^3}-\frac{1^2.3^2.5^2.7}{1^2.2^2.3^2.4^2}\frac{h^4}{4^4}-\&c.,$$

we find

$$c_1=1+\frac{3}{2},$$

$$c_2=c_1+\frac{1^2.3}{1^2.2^2}\left\{1+\frac{3.5}{3}\right\},$$

$$c_3=c_2+\frac{1^2.3}{1^2.2^2}c_1+\frac{1^2.3^2.5}{1^2.2^2.3^2}\left\{1+\frac{5.7}{4}\right\},$$

$$c_4=c_3+\frac{1^2.3}{1^2.2^2}c_2+\frac{1^2.3^2.5}{1^2.2^2.3^2}c_1=\frac{1^2.3^2.5^2.7}{1^2.2^2.3^2.4^2}\left\{1+\frac{7.9}{5}\right\},$$

$$\&c., \qquad \&c.,$$

giving the same values of c_1, c_2, ... as above.

§ 14. Since

$$-\frac{2I}{E}=-\frac{4h}{E}\frac{dE}{dh}=-4h\frac{d}{dh}\log E,$$

we find at once

$$\log E=-\frac{h}{4}-\frac{c_1}{2}\frac{h^2}{4^2}-\frac{c_2}{3}\frac{h^3}{4^3}-\frac{c_3}{4}\frac{h^4}{4^4}-\&c.,$$

and since

$$\frac{u^2}{4}=-\frac{h}{h'}+h\frac{du}{dh},$$

we find

$$\frac{I^2}{E^2}=(2c_1-4)\frac{h^2}{4}+(3c_2-4^2)\frac{h^3}{4^2}+(4c_3-4^3)\frac{h^4}{4^3}$$
$$+(5c_4-4^4)\frac{h^5}{4^4}+\&c.$$

§ 15. Let

$$u=-\frac{2E}{I},$$

then

$$\frac{du}{dh}=-\frac{1}{h}-\frac{u^2}{4h'}.$$

Putting

$$u = \frac{4}{h} + d_1 + d_2\frac{h}{4} + d_3\frac{h^2}{4^2} + d_4\frac{h^3}{4^3} + \&c.,$$

and substituting in the equation

$$(1-h)\frac{du}{dh} = 1 - \frac{1}{h} - \frac{u^2}{4},$$

we find

$$\begin{aligned} 2d_1 &= -5, \\ 3d_2 + d_1^2 &= 4, \\ 4d_3 + 2d_2d_1 &= 4d_2, \\ 5d_4 + 2d_3d_1 + d_2^2 &= 8d_3, \\ 6d_5 + 2d_4d_1 + 2d_3d_2 &= 12d_4, \\ \&c., \qquad & \&c. \end{aligned}$$

Change of h into $-\frac{h'}{h}$, § 17.

§ 17. By the change of h into $-\dfrac{h'}{h}$, we convert $\dfrac{G}{K}$ and $\dfrac{I}{E}$ into $\dfrac{I}{h'K}$ and $\dfrac{G}{E}$ respectively. If therefore in the above expansions we replace h by $-\dfrac{h'}{h}$, we obtain the expansions of

$$\frac{I}{h'K}, \quad \frac{h'K}{I}, \quad \frac{G}{E}, \quad \frac{E}{G},$$

in series having their terms alternately positive and negative.

General Remarks, §§ 18, 19.

§ 18. Neither of the methods employed in this paper afford interesting systems of equations for the determination of the coefficients. Perhaps other methods might give simpler and more elegant relations. It is evident that by means of the quotient method we may obtain directly the expansions of the ratios of any two of the quantities K, I, G, E.

§ 19. The investigation of the above developments was suggested by Mr. Childe's paper (pp. 155–164). Mr. Childe puts $F = (1+m)E$, so that his $m = -\frac{I}{E}$, and obtains an expression for m depending upon $\frac{h^2}{h'}$.

I may mention that Mr. F. W. Newman, in his "Elliptic Integrals" (1889), considers the ratio $\frac{E}{F}$, which he terms the 'ancilla.' He gives a formula (p. 28) which may be written

$$-\frac{I}{kK} = \tfrac{1}{2}k + \tfrac{1}{4}kk_1 + \tfrac{1}{8}kk_1k_2 + \tfrac{1}{16}kk_1k_2k_3 + \&c.,$$

where k_1, k_2, ... are successive moduli formed according to Landen's scale.

It should be added also that the differential equations satisfied by $\frac{G}{K}$, $\frac{K}{G}$, $\frac{I}{E}$, $\frac{E}{I}$ were given by Mr. Kleiber, in vol. XVIII., p. 176, of the *Messenger* (April, 1889), the modular angle θ being the independent variable.

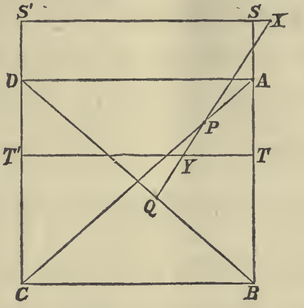
NOTE ON RECIPROCAL LINES.

By Prof. *Cayley*.

If two lines are reciprocal in regard to a quadric surface, then any point on the one line and any point on the other line are harmonics in regard to the surface (viz. the two points and the intersections of their line of junction with the surface form a harmonic range). This is obvious, the polar plane of the first point passes through the second line, and thus the second point is a point in the polar plane of the first point, that is, the two points are harmonics.

But it is worth while to look at the theorem in a different point of view: if the first line meets the surface in the points A, C and the second line meets the surface in the points B, D (in order to the four points being real, the surface must of course be a skew hyperboloid), then AB, BC, CD, DA are lines on the surface, say AB, CD are directrices, and BC, AD are generatrices, the two reciprocal lines being the diagonals AC and BD of the skew quadrilateral. Taking on AC a point P and on BD a point Q, the theorem is that if PQ meets the surface in the points X, Y, then the four points

P, Q, X, Y are harmonics. Let the generatrix through X meet AB, CD in the points S, S' respectively, and the generatrix through Y meet the same lines in the points T, T' respectively. Then the four lines CB, DA, $T'T$, $S'S$, are all met by an infinite series of lines, and in particular they are met by the lines CD, BA in the points C, D, T', S' and B, A, T, S respectively. Consequently, writing AH for anharmonic ratio, we have



$$AH(C, D, T', S') = AH(B, A, T, S).$$

Consider now the four lines CA, DB, $T'T$, $S'S$, these are each of them met by the lines CD and BA, and also by the line PQ; therefore, by an infinity of lines (that is, they are directrices of a hyperboloid), and they are met by the last-mentioned three lines in the points C, D, T', S'; A, B, T, S; and P, Q, Y, X respectively; hence

$$AH(C, D, T', S') = AH(A, B, T, S) = AH(P, Q, Y, X).$$

Comparing with the foregoing equation, it appears that

$$AH(B, A, T, S) = AH(A, B, T, S),$$

viz., the anharmonic ratio is not altered by the interchange of the two points A, B: this implies that the points A, B, T, S are harmonics, viz. the pairs A, B and S, T are harmonics; and, this being so, the equation

$$AH(A, B, T, S) = AH(P, Q, Y, X)$$

shows that P, Q, Y, X are harmonics, viz. that the pairs P, Q and X, Y are harmonics; or say P, Q are harmonics in regard to the quadric surface, which is the theorem in question.

A PROOF OF THE THEOREM OF RECIPROCITY FOR QUADRATIC RESIDUES.

By *F. Franklin*, Assistant Professor in the Johns Hopkins University.

1. It was shewn by Gauss that, p being an odd prime, $D^{\frac{1}{2}(p-1)} \equiv 1$ or -1 (mod. p) according as, in the series

$$D,\ 2D,\ 3D,\ \ldots,\ \tfrac{1}{2}(p-1)\,D,$$

the number of numbers whose least positive remainder (mod. p) exceeds $\frac{1}{2}p$ is even or odd. But to say that the least positive remainder of λD exceeds $\frac{1}{2}p$ is the same as to say that $E\dfrac{2\lambda D}{p}$ is an odd number. Hence we have the transformed criterion: *D is a quadratic residue or non-residue of p* according as the number of odd numbers in the series

$$E\frac{2D}{p},\ E\frac{4D}{p},\ \ldots,\ E\frac{(p-1)\,D}{p}$$

is even or odd: hence *according as*

$$E\frac{2D}{p} + E\frac{4D}{p} + \ldots + E\frac{(p-1)\,D}{p}$$

is even or odd.

2. *If a and b be any two relative primes,*

$$E\frac{a}{b} + E\frac{2a}{b} + E\frac{3a}{b} + \ldots + E\frac{(b-1)\,a}{b} = \tfrac{1}{2}(a-1)(b-1).$$

For, if we write under this series the same series reversed, the sum of the two complete fractions in any column is a; therefore, each being actually fractional, the sum of their integer parts is $a-1$; hence the sum of the proposed series is $\frac{1}{2}(a-1)(b-1)$.

3. Let us denote *the number of odd numbers* in a set by prefixing the symbol I (impar.); then, *if a and b are odd relative primes,*

$$I\left\{E\frac{2a}{b},\ E\frac{4a}{b},\ \ldots,\ E\frac{(b-1)\,a}{b}\right\}$$
$$= \tfrac{1}{2}I\left\{E\frac{a}{b},\ E\frac{2a}{b},\ E\frac{3a}{b},\ \ldots,\ E\frac{(b-1)\,a}{b}\right\}.$$

For, in the complete set, if any term is odd its symmetrical is odd (their sum being $a-1$, an even number); and if one of these terms belongs to the even set $\left(E\frac{2a}{b},\ E\frac{4a}{b},\ \dots\right)$, the other does not. Hence the even set contains half as many odd numbers as the complete set. Q.E.D.

4. By 2,

$$E\frac{D}{p}+E\frac{2D}{p}+E\frac{3D}{p}+E\frac{4D}{p}+\dots$$
$$+E\frac{(p-2)D}{p}+E\frac{(p-1)D}{p}=\tfrac{1}{2}(p-1)(D-1).$$

And if $D<p$, it is plain that

$$-E\frac{D}{p}+E\frac{2D}{p}-E\frac{3D}{p}+E\frac{4D}{p}-\dots-E\frac{(p-2)D}{p}+E\frac{(p-1)D}{p}$$
$$=I\left\{E\frac{p}{D},\ E\frac{2p}{D},\ E\frac{3p}{D},\ \dots,\ E\frac{(D-1)p}{D}\right\};$$

for $E\frac{2\lambda D}{p}-E\frac{(2\lambda-1)D}{p}$ is 1 or 0 according as there is or is not a multiple of p between $(2\lambda-1)D$ and $2\lambda D$; in other words, it is 1 as many times as there are multiples of p, not exceeding $(p-1)D$, whose quotient by D is an odd number; whence the above equation.

Adding the equations above written, we have

$$E\frac{2D}{p}+E\frac{4D}{p}+\dots+E\frac{(p-1)D}{p}$$
$$=\frac{p-1}{2}\,\frac{D-1}{2}+\tfrac{1}{2}I\left\{E\frac{p}{D},\ E\frac{2p}{D},\ \dots,\ E\frac{(D-1)p}{D}\right\}.$$

If D is odd, this becomes, by 3,

$$E\frac{2D}{p}+E\frac{4D}{p}+\dots+E\frac{(p-1)D}{p}$$
$$=\frac{p-1}{2}\,\frac{D-1}{2}+I\left\{E\frac{2p}{D},\ E\frac{4p}{D},\ \dots,\ E\frac{(D-1)p}{D}\right\}$$
$$\equiv\frac{p-1}{2}\,\frac{D-1}{2}+E\frac{2p}{D}+E\frac{4p}{D}+\dots+E\frac{(D-1)p}{D}\pmod{2}.$$

Hence, if D is an odd prime, the quadratic characters of D with respect to p and of p with respect to D are the same, unless $\frac{1}{2}(p-1)$ and $\frac{1}{2}(D-1)$ are both odd, in which case the characters are opposite.

NOTE DE GÉOMÉTRIE À PROPOS D'UN THÉORÈME DE M. STEWART.

Par M. le Prof. *A. Mannheim.*

Soient a, b, c, d les sommets d'un quadrilatère convexe inscrit dans une circonférence de cercle. Il est évident que la somme des aires des deux triangles dans lesquels il est partagé par l'une de ses diagonales est égale à la somme des aires des deux autres triangles déterminés par l'autre diagonale. Ainsi on peut écrire tout de suite

$$(1)\quad abc + acd = abd + bcd.$$

Exprimant l'aire de chacun de ces triangles en fonction du produit de leurs trois côtés et du rayon du cercle qui leur est circonscrit et remarquant que ce cercle est commun aux quatre triangles, on a simplement

$$(2)\quad ab.bc.ac + ac.cd.ad = ab.bd.ad + bc.cd.ad,$$

relation qui donne l'expression du rapport des diagonales du quadrilatère $abcd$ en fonction de ses côtés, et à laquelle on arrive ainsi d'une manière rapide et très simple.

Transformons par rayons vecteurs réciproques les sommets du quadrilatère $abcd$ et la circonférence circonscrite à ce quadrilatère en prenant pour pôle de transformation un point arbitraire O dans l'espace.

On obtient ainsi les points A, B, C, D qui sont encore les sommets d'un quadrilatère convexe inscrit dans une circonférence de cercle.

Appliquant une formule bien connue* à chacun des segments qui entrent dans la relation (2) celle ci devient :

$$\frac{AB}{OA.OB} \times \frac{BC}{OB.OC} \times \frac{AC}{OA.OC} + \frac{AC}{OA.OC} \times \frac{CD}{OC.OD} \times \frac{AD}{OA.OD}$$
$$= \frac{AB}{OA.OB} \times \frac{BD}{OB.OD} \times \frac{AD}{OA.OD} + \frac{BC}{OB.OC} \times \frac{CD}{OC.OD} \times \frac{BD}{OB.OD},$$

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* *Eléments de Géométrie*, par Rouché et de Comberousse, 4^e édition, page 367.

que l'on peut écrire:

$$(3)\quad AB.BC.AC(OD)^2+AC.CD.AD(OB)^2$$
$$=AB.BD.AD(OC)^2+BC.CD.BD(OA)^2.$$

Si l'on introduit les aires des triangles dans lesquels le quadrilatère $ABCD$ est partagé par ses diagonales, on a

$$ABC(OD)^2+ACD(OB)^2=ABD(OC)^2+BCD(OA)^2,$$

qui est la transformée par rayons vecteurs réciproques de la relation (1). Elle permet de retrouver celle ci lorsque O est à l'infini ou en un point de l'axe de la circonférence circonscrite au quadrilatère $ABCD$.

Reprenons la relation (3) relative à quatre points qui se suivent sur une circonférence de cercle et un point arbitraire O dans l'espace.

Lorsque le rayon de la circonférence $ABCD$ est infini la relation (3) concerne quatre points d'une droite et un point O arbitraire.

Si le point O est lui-même sur la droite $ABCD$ la relation (3), qui s'applique toujours, concerne cinq points en ligne droite.

Cette dernière relation a été donnée sans démonstration par Chasles à la page 176 de son *Aperçu historique.* Chasles a donné en outre des cas particuliers de cette relation concernant quatre points d'une droite, cas particuliers que je ne crois pas utile de reproduire ici et qu'il est du reste facile de retrouver.

Replaçons le point O en dehors de la droite $ABCD$ et écrivons ainsi la relations (3)

$$AB.BC.AC+AC(OB)^2\frac{CD.AD}{(OD)^2}$$
$$=AB(OC)^2\frac{BD.AD}{(OD)^2}+BC(OA)^2\frac{CD.BD}{(OD)^2}.$$

Si l'on suppose maintenant le point D à l'infini il vient simplement:

$$AB.BC.AC+AC(OB)^2=AB(OC)^2+BC(OA)^2,$$

qui est l'expression du théorème de Stewart.

Ainsi le théorème de Stewart résulte ici de la transformation par rayons vecteurs réciproques du théorème de géométrie élémentaire qui donne l'expression du rapport des diagonales d'un quadrilatère convexe inscrit dans une circonférence de cercle. Cette transformation peut s'effectuer directement en plaçant le pôle de transformation en un des sommets du quadrilatère.

La transformation de la relation (2) a donné la relation (3). Il ne faut pas croire qu'en transformant cette dernière relation on soit conduit à en obtenir une nouvelle concernant six points.

En opérant ainsi on ne fera que retrouver la relation (3).

Ceci est facile à vérifier. Mais cette vérification est inutile car l'on sait, en vertu d'une belle proposition due à Liouville que, lorsqu'il s'agit de la transformation par rayons vecteurs réciproques, le résultat de transformations successives d'une propriété ne fait que donner le résultat obtenu directement par une seule transformation générale de cette propriété.

Remarques. Appelons s l'aire du triangle abc, S l'aire du triangle ABC et k^2 la puissance de la transformation. Supposons O sur le plan de abc et désignons par T la longueur de la tangente menée du point O à la circonférence ABC.

Il est facile de voir, d'après ce qui précède, que:

$$s = \frac{S.T^2.k^4}{(OA)^2(OB)^2(OC)^2}.$$

On trouvera aussi très simplement cette autre expression:

$$2s = k^4\left(\frac{\sin COA}{OA.OC} + \frac{\sin BOC}{OB.OC} - \frac{\sin BOA}{OA.OB}\right).$$

Si en faisant usage de cette dernière expression, on transforme cette propriété: *les triangles de même aire qui ont pour base ab ont leur troisième sommet sur une parallèle à ab*, on trouve que: *Etant donnés trois points fixes A, B, O si l'on prend un point C, tel que*

$$\frac{\sin COA}{OA.OC} + \frac{\sin BOC}{OB.OC} = \text{const}^{\text{e}}.$$

le lieu des points tels que C est une circonférence tangente en O au cercle circonscrit au triangle ABO.

Il est évidemment très facile d'appliquer les formules précédentes à la transformation de théorèmes concernant les aires des triangles.

ON THE STABILITY OF A BENT AND TWISTED WIRE.

By *J. H. Michell*, Trinity College, Cambridge.

If a wire of isotropic section and naturally straight be twisted, and the two ends joined so as to form a continuous curve, the circle will be a stable form of equilibrium for less than a certain amount of twist.*

I propose in this note to determine the limit of stability. I begin by finding the general intrinsic equations of vibration of a bent wire.

Let AB be an element of the wire bounded by normal sections A, B, and let the distances of these sections from a fixed point of the wire be $s-\delta s$ and s respectively.

Let S, T, U be the components of the resultant force on the section B, S being measured along the tangent in the direction of s increasing, T along the principal normal inwards, and U along the binormal, so that the three directions form a right-handed system.

Let F, G, H be the components of the couple on the section B in the same three directions.

Then, if P, Q, R are the impressed forces on the element AB per unit length, the equations of equilibrium are

$$\left.\begin{aligned}
\frac{dS}{ds}-\kappa T+\quad P &= 0\\
\frac{dT}{ds}-\tau U+\kappa S+Q &= 0\\
\frac{dU}{ds}+\tau T+\quad R &= 0\\
\frac{dF}{ds}-\kappa G\qquad\qquad &= 0\\
\frac{dG}{ds}-\tau H+\kappa F-U &= 0\\
\frac{dH}{ds}+\tau G+\quad T &= 0
\end{aligned}\right\}\ldots\ldots\ldots\ldots\ldots(1),$$

where κ is the curvature and τ the torsion at s.

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* Thomson and Tait, *Nat. Phil.*, § 123.

Now let γ be the rate of twist of the wire at s, then the theory of wires gives

$$\left.\begin{aligned} F &= M\gamma \\ G &= 0 \\ H &= L\kappa \end{aligned}\right\},$$

assuming the flexibility the same in all directions.

Substituting in equations (1), we get

$$\frac{dF}{ds} = 0,$$

$$U = -L\,\kappa\tau + M\,\kappa\gamma,$$

$$T = -L\frac{d\kappa}{ds},$$

so that the twist γ is constant, and

$$S = -\frac{Q}{\kappa} - \tau\,(L\tau - M\gamma) + L\frac{1}{\kappa}\frac{d^2\kappa}{ds^2}.$$

Substituting in the two remaining equations of (1), we get the two dynamical equations

$$\left.\begin{aligned} L\frac{d}{ds}\left(\frac{1}{\kappa}\frac{d^2\kappa}{ds^2} + \tfrac{1}{2}\kappa^2 - \tau^2\right) + M\gamma\frac{d\tau}{ds} &= -P + \frac{d}{ds}\frac{Q}{\kappa} \\ L\left(\frac{d}{ds}\kappa\tau + \tau\frac{d\kappa}{ds}\right) - M\gamma\frac{d\kappa}{ds} &= R \end{aligned}\right\}\ldots(2).$$

Now let the wire vibrate about its equilibrium form, and let u, v, w be the displacements of the point s along the tangent, principal normal, and binormal respectively at time t.

Supposing no impressed forces we have

$$-P = m\frac{d^2u}{dt^2},$$

$$-Q = m\frac{d^2v}{dt^2},$$

$$-R = m\frac{d^2w}{dt^2},$$

and the condition of inextensibility is

$$\frac{du}{ds} = \kappa v,$$

so that
$$-Q = \frac{m}{\kappa}\frac{d^3u}{ds\,dt^2}.$$

Further, if κ_0, τ_0 denote the equilibrium values of the curvature and torsion respectively, we have*

$$\left.\begin{aligned} \kappa-\kappa_0 &= \frac{d\alpha}{ds}-\tau\beta \\ \tau-\tau_0 &= \frac{d}{ds}\,\frac{1}{\kappa}\,\frac{d\beta}{ds}+\kappa\beta+\frac{d}{ds}\,\frac{\tau}{\kappa}\,\alpha \end{aligned}\right\}\ldots\ldots\ldots\ldots(3),$$

where
$$\alpha=\frac{d}{ds}\,\frac{1}{\kappa}\,\frac{du}{ds}+\kappa u-\tau w,$$

$$\beta=\frac{dw}{ds}+\frac{\tau}{\kappa}\,\frac{du}{ds}.$$

Substituting these values in equations (2), we have the general intrinsic equations of vibration.

When the equilibrium form is a plane curve, these equations reduce to

$$\left.\begin{aligned} &L\frac{d}{ds}\left(\frac{1}{\kappa}\,\frac{d^2\kappa}{ds^2}+\tfrac{1}{2}\kappa^2\right)+M\gamma\frac{d\tau}{ds}=m\frac{d^2u}{dt^2}-m\frac{d}{ds}\,\frac{1}{\kappa^2}\,\frac{d^3u}{ds\,dt^2} \\ &L\left(\frac{d}{ds}\,\kappa\tau+\tau\frac{d\kappa}{ds}\right)-M\gamma\frac{d\kappa}{ds}=-\,m\frac{d^2w}{dt^2} \end{aligned}\right\},$$

where
$$\kappa=\kappa_0+\frac{d^2}{ds^2}\,\frac{1}{\kappa}\,\frac{du}{ds}+\frac{d}{ds}\,\kappa u,$$

$$\tau=\frac{d}{ds}\,\frac{1}{\kappa}\,\frac{d^2w}{ds^2}+\kappa\frac{dw}{ds}.$$

Proceeding to the particular case of a circular ring, the equations are

$$L\frac{1}{\kappa^2}\left(\frac{d^3}{ds^3}+\kappa^2\frac{d}{ds}\right)^2u+M\gamma\frac{1}{\kappa}\left(\frac{d^4}{ds^4}+\kappa^2\frac{d^2}{ds^2}\right)w=m\left(\frac{d^2u}{dt^2}-\frac{1}{\kappa^2}\frac{d^4u}{ds^2dt^2}\right),$$

$$-L\left(\frac{d^4}{ds^4}+\kappa^2\frac{d^2}{ds^2}\right)w+M\gamma\frac{1}{\kappa}\left(\frac{d^4}{ds^4}+\kappa^2\frac{d^2}{ds^2}\right)u=m\frac{d^2w}{dt^2}.$$

The appropriate solution, when the wire forms a complete circle, is

$$u=Ae^{i(pt-rs\kappa)},$$
$$w=Be^{i(pt-rs\kappa)},$$

r being an integer.

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* "The small deformation of curves and surfaces, &c.," *ante* p. 68.

Making the substitutions and eliminating A, B, we find

$$\begin{vmatrix} mp^2(1+r^2)-L\kappa^4r^2(1-r^2)^2, & -M\gamma\kappa^3r^2(1-r^2) \\ -M\gamma\kappa^3r^2(1-r^2), & mp^2+L\kappa^4r^2(1-r^2) \end{vmatrix}=0,$$

or

$$m^2p^4(1+r^2)+2mp^2L\kappa^4r^4(1-r^2)-L^2\kappa^8r^4(1-r^2)^3$$
$$-M^2\gamma^2\kappa^6r^4(1-r^2)^2=0.$$

For stability, the values of p^2 must be positive, and this leads to the condition

$$L^2\kappa^2(r^2-1)>M^2\gamma^2.$$

Now $r=1$ corresponds merely to displacement of the ring as a rigid body.

The necessary condition for stability is therefore

$$\frac{\gamma}{\kappa}<\frac{L}{M}\sqrt{(3)},$$

so that the total twist must be less than

$$2\sqrt{(3)}\,\pi\,L/M.$$

If the cross-section is circular,

$$\frac{L}{M}=\frac{E}{2\mu},$$

where E is Young's modulus and μ is the rigidity modulus.

For metals $E=\frac{5}{2}\mu$ about, and in this case the total twist must be less than $2\pi\times 2.16$.

ON THE EQUATION $x^{17}-1=0$.

By Prof. *Cayley*.

WRITING $\rho=\cos\frac{2\pi}{17}+i\sin\frac{2\pi}{7}$, I carry the solution up to the determination of the periods each of two roots, $\rho+\rho^{16}$, $=2\cos\frac{2\pi}{17}$, &c. The expressions contain the radicals

$$a=\sqrt{(17)},\ b=\sqrt{\{2(17-a)\}},\ c=\sqrt{\{4(17+3a)-2(3+a)b\}},$$

where a, b, c are taken to be positive ($a=4{\cdot}12$, $b=5{\cdot}07$, $c=6{\cdot}72$). Taking for a moment r to be any imaginary seventeenth root, $r=\rho^6$, then the algebraical expression for the period P_1 of eight roots is $P_1=\frac{1}{2}(-1\pm a)$, but I assume the value to be $P_1=\frac{1}{2}(-1+a)$, and thus determine θ to denote some one of the values 1, 2, 4, 8, 9, 13, 15, 16; similarly, I assume the value of Q_1 to be $=\frac{1}{4}(-1+a+b)$; and thus further determine θ to denote some one of the values 1, 4, 13, 16: and, again, I assume the value of R_1 to be $=\frac{1}{8}(-1+a+b+c)$, and thus further determine θ to denote one of the values 1 and 16. As regards the values of the periods R, it is obviously indifferent which value is taken, and I assume therefore $\theta=1$. This comes to saying that the signs of the radicals are determined in suchwise that r shall denote the root $\cos\frac{2\pi}{17}+i\sin\frac{2\pi}{17}$, and it is to be understood that r has this value.

I write now

$$
\begin{aligned}
P_1 &= r + r^9 + r^{13} + r^{15} + r^{16} + r^8 + r^4 + r^2,\\
P_2 &= r^3 + r^{10} + r^5 + r^{11} + r^{14} + r^7 + r^{12} + r^6,\\
Q_1 &= r + r^{13} + r^{16} + r^4,\\
Q_2 &= r^9 + r^{15} + r^8 + r^2,\\
Q_3 &= r^3 + r^5 + r^{14} + r^{12},\\
Q_4 &= r^{10} + r^{11} + r^7 + r^6,\\
R_1 &= r + r^{16},\\
R_2 &= r^{13} + r^4,\\
R_3 &= r^9 + r^8,\\
R_4 &= r^{15} + r^2,\\
R_5 &= r^3 + r^{14},\\
R_6 &= r^5 + r^{12},\\
R_7 &= r^{10} + r^7,\\
R_8 &= r^{11} + r^6,
\end{aligned}
$$

and moreover

$$
\begin{aligned}
a &= P_1 - P_2,\\
b &= 2(Q_1 - Q_2),\\
+b_1 &= 2(Q_3 - Q_4),\\
c &= 4(R_1 - R_2),\\
-c_1 &= 4(R_3 - R_4),\\
+c_2 &= 4(R_5 - R_6),\\
-c_3 &= 4(R_7 - R_8).
\end{aligned}
$$

It will appear by what follows that a is determined by the quadric equation $a^2=17$, but I have assumed that a denotes the positive root $a=\sqrt{(17)}$; similarly b is determined by the quadric equation $b^2=2\,(17-a)$, but it is assumed that b denotes the positive root, $b=\sqrt{\{2\,(17-a)\}}$; and c is determined by the quadric equation $c^2=4\,(17+3a)-2\,(3+a)\,b$, but it is assumed that c denotes the positive root,

$$=\sqrt{\{4\,(17+3a)-2\,(3+a)\,b\}}.$$

If in the equations I had written b_1, c_1, c_2, c_3 (instead of $+b_1$, $-c_1$, $+c_2$, $-c_3$), then b_1 comes out rationally in terms of a, b; and c_1, c_2, c_3 come out rationally in terms of a, b, c; the signs were attached to them *à posteriori*, in suchwise that the values of b_1, c_1, c_2, c_3 might be each of them positive; for their independent determination, we have in fact for b_1^2 an expression such as that for b^2; and for c_1^2, c_2^2, c_3^2 expressions such as that for c^2; and taking as above for each of them the positive value of the square root, we have

$$\begin{array}{ll}
a=\sqrt{(17)}, & (=\ 4{\cdot}12),\\
b=\sqrt{\{2\,(17-a)\}}, & (=\ 5{\cdot}07),\\
b_1=\sqrt{\{2\,(17+a)\}}, & (=\ 6{\cdot}49),\\
c=\sqrt{\{4\,(17+3a)-2\,(\ \ 3+a)\,b\}}, & (=\ 6{\cdot}72),\\
c_1=\sqrt{\{4\,(17+3a)+2\,(\ \ 3+a)\,b\}}, & (=13{\cdot}77),\\
c_2=\sqrt{\{4\,(17-3a)+2\,(-3+a)\,b_1\}}, & (=\ 5{\cdot}75),\\
c_3=\sqrt{\{4\,(17-3a)-2\,(-3+a)\,b_1\}}, & (=\ 2{\cdot}02).
\end{array}$$

The relations between the periods P are

$$P_1+P_2=-1,$$

	P_1	P_2
P_1	$-P_1+4$	-4
P_2		$-P_2+4$

	a
a	17

that is $P_1^2=-P_1+4$, &c.; $a^2=17$.

Here for the second square we have

$$a^2=(P_1-P_2)^2=P_1^2+P_2^2-2P_1P_2=(-P_1+4)+(-P_2+4)+8,$$
$$=-P_1-P_2+16,=17;$$

and similarly in the other cases which follow.

For the periods Q, we have

$$Q_1+Q_2=P_1,$$
$$Q_3+Q_4=P_2,$$

	Q_1	Q_2	Q_3	Q_4
Q_1	Q_2+2Q_3+4	-1	$2Q_1+Q_2+Q_4$	$Q_2+Q_3+2Q_4$
Q_2		Q_1+2Q_4+4	Q_1+2Q_3+4	$Q_1+2Q_2+Q_3$
Q_3			$2Q_2+Q_4+4$	-1
Q_4				$2Q_1+Q_3+4$

	b	b_1
b	$34-2a$	$8a$
b_1		$34+2a$

so that $bb_1=8a$, or b_1 is given rationally in terms of a, b.

And for the periods R, we have

$$R_1+R_2=Q_1,$$
$$R_3+R_4=Q_2,$$
$$R_5+R_6=Q_3,$$
$$R_7+R_8=Q_4,$$

	R_1	R_2	R_3	R_4	R_5	R_6	R_7	R_8
R_1	R_4+2	$\overline{R_5+R_6}$	R_3+R_7	R_1+R_5	R_2+R_4	R_2+R_8	R_3+R_8	R_6+R_7
R_2		R_3+2	R_2+R_6	R_4+R_8	R_1+R_7	R_1+R_3	R_5+R_8	R_4+R_7
R_3			R_1+2	$\overline{R_7+R_8}$	R_6+R_8	R_2+R_5	R_1+R_4	R_4+R_5
R_4				R_2+2	R_1+R_6	R_5+R_7	R_3+R_8	R_2+R_3
R_5					R_8+2	$\overline{R_3+R_4}$	R_2+R_7	R_3+R_5
R_6						R_7+2	R_4+R_6	R_1+R_8
R_7							R_5+2	$\overline{R_1+R_2}$
R_8								R_3+2

where observe that the overlined terms R_5+R_6, R_7+R_8, R_3+R_4, and R_1+R_2, have the values Q_4, Q_3, Q_2, Q_1, respectively, and

	c	c_1	c_2	c_3
c	$4(17+3a)-2(3+a)b$	$8(b+b_1)$	$4(2a-b+b_1)$	$4(-2a+b+b_1)$
c_1		$4(17+3a)+2(3+a)b$	$4(2a+b+b_1)$	$4(2a+b-b_1)$
c_2			$4(17-3a)+2(-3+a)b_1$	$8(-b+b_1)$
c_3				$4(17-3a)-2(-3+a)b_1$

in which last table b_1 may be considered as denoting its value $=\frac{8a}{b}$, so that c_1, c_2, c_3 are each given rationally in terms of a, b, c.

And from the foregoing results, we have

$$\begin{aligned}
P_1 &= \tfrac{1}{2}(-1+a), & &= 1\cdot 56,\\
P_2 &= \tfrac{1}{2}(-1-a), & &= -2\cdot 56,\\
Q_1 &= \tfrac{1}{4}(-1+a+b), & &= 2\cdot 05,\\
Q_2 &= \tfrac{1}{4}(-1+a-b), & &= -0\cdot 49,\\
Q_3 &= \tfrac{1}{4}(-1-a+b_1), & &= 0\cdot 34,\\
Q_4 &= \tfrac{1}{4}(-1-a-b_1), & &= -2\cdot 90,\\
R_1 &= \tfrac{1}{8}(-1+a+b+c), & &= 1\cdot 87,\\
R_2 &= \tfrac{1}{8}(-1+a+b-c), & &= 0\cdot 18,\\
R_3 &= \tfrac{1}{8}(-1+a-b-c_1), & &= -1\cdot 96,\\
R_4 &= \tfrac{1}{8}(-1+a-b+c_1), & &= 1\cdot 47,\\
R_5 &= \tfrac{1}{8}(-1-a+b_1+c_2), & &= 0\cdot 8\ ,\\
R_6 &= \tfrac{1}{8}(-1-a+b_1-c_2), & &= -0.55,\\
R_7 &= \tfrac{1}{8}(-1-a-b_1-c_3), & &= -1\cdot 70,\\
R_8 &= \tfrac{1}{8}(-1-a-b_1+c_3), & &= -1\cdot 20.
\end{aligned}$$

The approximate numerical values have been given throughout only for the purpose of showing that the signs of the square roots have been rightly determined.

QUATERNION PROOFS OF THEOREMS RELATING TO ASYMPTOTIC LINES.

By *T. Motoda.*

I HAVE proved in a simple way some theorems in Geometry, concerning asymptotic lines, by the use of the quaternion analysis as follows:

Let ρ be the vector of an asymptotic curve of the surface $S\nu d\sigma=0$; then $S\nu d\rho=0$, or ν is perpendicular to $d\rho$. Along the curve two consecutive normals of the surface are parallel, or $V\nu(\nu+d\nu)=0$ or $V\nu d\nu=0$, or ν is parallel to $d\nu$. But ν is perpendicular to $d\rho$, hence $d\nu$ is perpendicular to $d\rho$ or $Sd\nu d\rho=0$, which is the required equation of the curve. Differentiating the equation $S\nu d\rho=0$ we obtain $Sd\nu d\rho+S\nu d^2\rho=0$; but along the curve the first term is zero, whence $S\nu d^2\rho=0$, which is another equation of the curve. Now $d^2\rho$ is the binormal of the curve, and the last equation shows that this is perpendicular to the normal ν of the surface; that is, the osculating plane of the asymptotic line is the tangent plane to the surface at this point.

The equation for the reciprocal of a surface is obtained by interchanging ν and ρ in the equation of the original surface. But the equation $Sd\nu d\rho=0$ is symmetrical in ν and ρ, hence the reciprocal of an asymptotic line of a surface is an asymptotic line of the reciprocal surface. If the surface is self-reciprocal, its asymptotic lines are also self-reciprocal. In this case any of such curves may be considered as the intersection of two of such surfaces, which are infinitely near; hence the curve is algebraical, and its degree (= class) is twice the degree (= class) of the surface. For example, the asymptotic lines of the sixteen nodal quartic surfaces are of the eighth class and of the eighth degree, a theorem of Klein and Lie.

Tokio, Japan,
February 5, 1890.

SOME INEQUALITIES.

By *Hugh W. Segar*, Trinity College, Cambridge.

I. It is well known that if H_n denote the sum of the homogeneous products of a, b, c of the nth degree we have

$$H_n=\frac{a^{n+2}(b-c)+b^{n+2}(c-a)+c^{n+2}(a-b)}{(a-b)(b-c)(a-c)}.$$

If then we consider a, b, c all positive, and suppose

$$\left.\begin{array}{ll} & a>b>c \\ \text{or} & b>c>a \\ \text{or} & c>a>b \end{array}\right\}\ldots\ldots\ldots\ldots\ldots\ldots(1),$$

we have $a^nb+b^nc+c^na>ab^n+bc^n+ca^n$,

when n is any positive integer.

We shall show generally that, under these conditions (1),

$$a^mb^n+b^mc^n+c^ma^n>a^nb^m+b^nc^m+c^na^m,$$

if m, n be any positive quantities such that $m>n$.

Scott in his *Theory of Determinants* gives

$$\begin{vmatrix} a^\alpha, & a^\beta, & a^\gamma \\ b^\alpha, & b^\beta, & b^\gamma \\ c^\alpha, & c^\beta, & c^\gamma \end{vmatrix} = (b-c)(c-a)(a-b) \begin{vmatrix} H_\alpha, & H_\beta, & H_\gamma \\ H_{\alpha-1}, & H_{\beta-1}, & H_{\gamma-1} \\ H_{\alpha-2}, & H_{\beta-2}, & H_{\gamma-2} \end{vmatrix}.$$

Using the relation between homogeneous functions of successive degrees, this may be made to yield

$$\Sigma a^m (b^n - c^n) = (a-b)(b-c)(c-a) \begin{vmatrix} H_{m-1}, & H_{n-1} \\ H_{m-2}, & H_{n-2} \end{vmatrix},$$

m, n being positive integers.

Now $$\Sigma (a^m b - ab^m) > 0,$$

gives $$\Sigma (a^m b^{m-1} - a^{m-1} b^m) > 0.$$

Putting then $m-1$ for n, we get

$$\begin{vmatrix} H_{m-1}, & H_{m-2} \\ H_{m-2}, & H_{m-3} \end{vmatrix} < 0,$$

or $$H^2_{m-2} > H_{m-1} H_{m-3}.$$

It follows that

$$\frac{H_0}{H_1} < \frac{H_1}{H_2} < \frac{H_2}{H_3} < \frac{H_3}{H_4} \ldots\ldots,$$

and $$H_r H_s < H_{r+1} H_{s-1} < H_{r+2} H_{s-2} \ldots\ldots,$$

if $$r < s.$$

Hence when m, n are positive integers, such that $m < n$, we have

$$\Sigma a^m (b^n - c^n) > 0,$$

and since a, b, c may be replaced by any of their powers or roots this must also be true for fractional values of m, n.

II. Put $m = n + t$, where t is small.

The finite terms cancel and the inequality may be made to depend on the coefficient of t.

We thus get

$$\Sigma \frac{\log b - \log c}{a} > 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (2),$$

i.e. $$\left(\frac{b}{c}\right)^{\frac{1}{a}} \left(\frac{c}{a}\right)^{\frac{1}{b}} \left(\frac{a}{b}\right)^{\frac{1}{c}} > 1.$$

Replacing a, b, c by their reciprocals, we get

$$\left(\frac{b}{c}\right)^{a}\left(\frac{c}{a}\right)^{b}\left(\frac{a}{b}\right)^{c} > 1.$$

Putting e^a, e^b, e^c for a, b, c, we get

$$(a-b)\,e^{a+b} + (b-c)\,e^{b+c} + (c-a)\,e^{c+a} > 0,$$

and $$(b-c)\,e^{a} + (c-a)\,e^{b} + (a-b)\,e^{c} > 0.$$

In the theorem of the arithmetical and geometrical means, take ab^m, bc^m, ca^m quantities respectively equal to

$$\left(\frac{a}{b}\right)^{m-1},\ \left(\frac{b}{c}\right)^{m-1},\ \left(\frac{c}{a}\right)^{m-1}, \text{ where } m > 1.$$

Then since $$\Sigma\,(a^m b - ab^m) > 0,$$

we get $$\left(\frac{b}{a}\right)^{ab^m}\left(\frac{c}{b}\right)^{bc^m}\left(\frac{a}{c}\right)^{ca^m} > 1.$$

or more generally

$$\left(\frac{b}{a}\right)^{b^m a^n}\left(\frac{c}{b}\right)^{c^m b^n}\left(\frac{a}{c}\right)^{a^m c^n} > 1.$$

where $$m \geqq n.$$

If we suppose a, b, c to be the sides of a triangle, we may replace them by $-a+b+c$, $a-b+c$, $a+b-c$, and reverse the inequality.

Now (2) gives us

$$a\log\left(\frac{b}{c}\right) + b\log\left(\frac{c}{a}\right) + c\log\frac{a}{b} > 0.$$

and making this substitution, we get

$$\Sigma\, a\log\frac{a-b+c}{a+b-c} > 0.$$

whence $$\left(\frac{\cos\frac{1}{2}B}{\cos\frac{1}{2}C}\right)^{\sin A}\left(\frac{\cos\frac{1}{2}C}{\cos\frac{1}{2}A}\right)^{\sin B}\left(\frac{\cos\frac{1}{2}A}{\cos\frac{1}{2}B}\right)^{\sin C} > 1.$$

Similarly from our original theorem we may derive

$$\Sigma \cot^r\tfrac{1}{2}B\,\{\tan^s\tfrac{1}{2}C - \tan^s\tfrac{1}{2}A\} > 0,$$

if r, s be positive and unequal.

III. The property of homogeneous functions, which we have shown in paragraph (1), is also true when there are more than three quantities.

For, if $h_0, h_1, h_2, \dots,$ be the homogeneous products of $b, c, d, \dots,$ and $H_0, H_1, H_2, \dots,$ those of $a, b, c, d, \dots,$ we have

$$H_n = \{a^n + a^{n-1}h_1 + \dots + h_n\} = K \quad \text{(say)}.$$

Then to prove our theorem by induction, we have only to show that

$$K^2 > \left\{\frac{K}{a} - \frac{h_n}{a}\right\}\{Ka + h_{n+1}\},$$

i.e. $$K^2 > K^2 + K\left(\frac{h_{n+1}}{a} - h_n\right) - \frac{h_n h_{n+1}}{a},$$

i.e. $h_n h_{n+1} + a h_n h_n + a^2 h_n h_{n-1} + \dots > h_n h_{n+1} + a h_{n+1} h_{n-1} + a^2 h_{n+1} h_{n-2} + \dots,$ and this follows if we assume the theorem to hold for $b, c, d, \dots$.

IV. A similar property characterizes the convergents $p_1, p_2, \dots,$ to a continued fraction x of the first class. Let them be arranged thus

$$p_1 p_3 p_5 \dots\dots x \dots\dots p_6 p_4 p_2,$$

in descending order of magnitude.

Then $$\frac{p_1}{p_3} > \frac{p_4}{p_2} > \frac{p_3}{p_5} > \frac{p_6}{p_4} > \dots.$$

Again if the equation

$$x^n - p_1 x^{n-1} + p_2 x^{n-2} - \dots (-1)^n p_n = 0$$

have all its roots real and positive, we have

$$p_1 > \frac{p_2}{p_1} > \frac{p_3}{p_2} > \dots.$$

Lastly, let s_r denote the sum of the rth powers of n positive quantities, we have then

$$\frac{s_1}{s_2} > \frac{s_2}{s_3} > \frac{s_3}{s_4} > \dots.$$

END OF VOL. XIX.

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